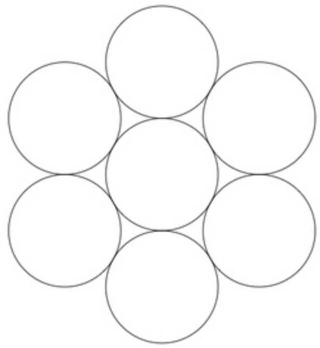
Seven Circles III

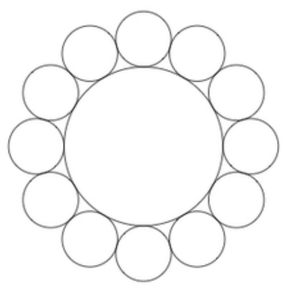
Task

Seven circles of the same size can be placed in the pattern shown below:

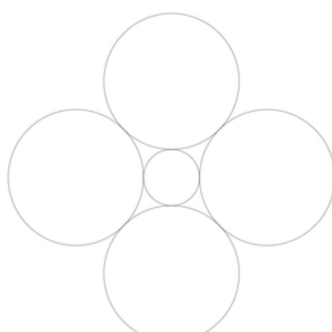


The six outer circles touch the one in the center and each circle on the outside also touches its two neighbors in the outside ring.

a. If twelve circles are placed around a central circle, as pictured below, what is the relationship between the diameters of the outer circles and the diameter of the inner circle? Explain.



b. If four circles are placed around a central circle, as pictured below, what is the relationship between the diameter of the outer circles and the diameter of the inner circle? Explain.



Commentary

This task is intended for instructional purposes only. It provides an opportunity to model a concrete situation with mathematics. Once a representative picture of the situation described in the problem is drawn (the teacher may provide guidance here as necessary), the solution of the task requires an understanding of the definition of the sine function. When the task is complete, new insight is shed on the ''Seven Circles I'' problem which initiated this investigation as is noted at the end of the solution.

This problem leads to what could make for a fun in class activity, namely measuring the diameters of different size coins (pennies, nickels, dimes, quarters, and half dollars) and using this information to estimate how many of each sized coin would fit around the circumference of a given coin. This information will be recorded in a separate task, ''Coins in a circular pattern.''

In the solution to both parts of this problem, the trigonometric function $sinx$ needs to be evaluated, once for the benchmark angle of 45 degrees and the other time for an angle of 15 degrees. While not a benchmark angle, the value for $\mathbf{sin(15)}$ can be found using a double angle formula. In the second solution, the values for $\mathbf{sin}\,\boldsymbol{x}$ are evaluated explicitly while in the first solution a calculator is used to find an approximate value.

Solutions

Solution: 2 Double angle formula

a. In order to calculate $\sin 15°$ exactly we may use the double angle formula for the cosine which is

$$\cos 2x = 1 - 2\sin^2 x\,.$$

Plugging in $x = 15°$ we find

$$1 - 2\sin^2 15° = \cos 30° = \frac{\sqrt{3}}{2}.$$

Solving for $\sin 15°$ we find

$$\sin 15° = \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}$$
$$= \frac{\sqrt{2-3}}{2}.$$

This exact expression can then be used to find the exact relationship between s and r since we know from the first solution that

$$s = \left(\frac{\sin 15°}{1 - \sin 15°}\right) r.$$

Simplifying this expression any further is rather tedious, but one ends with, among other possible expressions, the answer

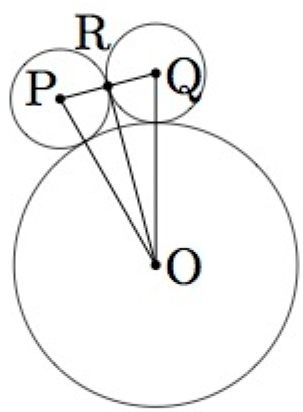
$$s = \left(\frac{\sqrt{3}-1}{1+2\sqrt{2}-\sqrt{3}}\right)r.$$

b. For part b finding and exact value for t is simpler than for part (a) because $\sin 45° = \frac{\sqrt{2}}{2}$. Using this we find

$$\begin{aligned} t &= \left(\frac{\sin 45°}{1-\sin 45°}\right)r \\ &= \left(\frac{\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right)r \\ &= (\sqrt{2}+1)r. \end{aligned}$$

Solution: 1

a. Let O be the center of the central circle and P, Q centers of adjacent circles in the ring and R the midpoint of segment PQ as pictured below:



Let r be the radius of the inner circle and s the radii of the twelve outer circles. Segment PO is congruent to segment QO as both have a length of $s+r$. Segment PR is congruent to segment QR because R is the midpoint of PQ. Finally segment OR is congruent to segment OR. By SSS we conclude that triangle POR is congruent to triangle QOR. Since angles ORP and ORQ are congruent and, taken together, add up to 180 degrees they are both right angles. Because there are twelve equal circles in the ring, the measure of angle POQ is one twelfth of 360 degrees or 30 degrees. Angles POR and QOR are congruent and so they much each be 15 degrees.

The sine of angle POR is the length of the side opposite the angle, $|PR|$ divided by the length of the hypotenuse $|OP|$. So we have

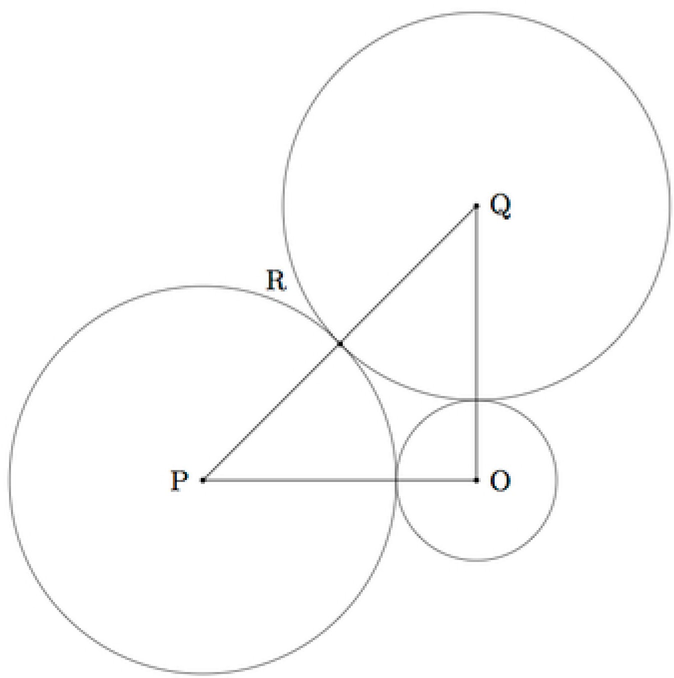
$$\sin(15°) = \frac{|PR|}{|OP|}$$
$$= \frac{s}{r+s}.$$

Multiplying both sides by $r + s$ gives $r \sin 15° + s \sin 15° = s$. Solving for s in terms of r we find

$$s = \left(\frac{\sin 15°}{1 - \sin 15°}\right) r.$$

For students who know double angle formulas, **sin 15** can be calculated directly as in the second solution below. Evaluating on a calculator, we find that s is about $0.349r$: as the picture shows, the circles in the ring are substantially smaller than the inner circle. For part (b) below, we should find, if the picture is accurate, that s is substantially larger than r. The case where $s = r$ was seen in "Seven Circles I."

b. The method used in part (a) works more generally as we will verify here for the case where there are four circles in the ring. We continue to denote the radius of the inner circle by r but use t this time for the radius of the outer circle. Choosing P and Q to be centers of adjacent circles in the outer ring and R the midpoint of segment PQ we find the following picture:



The remainder of the argument from part (a) now applies, the only change being that the measure of angle POR is now 45 degrees instead of 15 degrees. The sine of angle POR is the length of the side opposite the angle, $|PR|$ divided by the length of the hypotenuse $|OP|$. So we have

$$\sin(45°) = \frac{|PR|}{|OP|}$$
$$= \frac{t}{r+t}.$$

Multiplying both sides by $r + t$ gives $r \sin 45° + t \sin 45° = t$. Solving for t in terms of r we find

$$t = \left(\frac{\sin 45°}{1 - \sin 45°}\right) r.$$

Here students may use the fact that $\sin 45 = \frac{\sqrt{2}}{2}$ as is done in the second solution below. Evaluating on a calculator, we find that t is about $2.41r$, validating the picture which shows that the circles in the outer ring are substantially larger than the central circle.

Now that all of this work has been done we can check, as was asked in ''Seven Circles I'' that if six circles of radius r' are places around a circle of radius r in the pattern shown at the beginning of the problem, then

$$r' = \left(\frac{\sin 30°}{1 - \sin 30°}\right) r = r$$

so that all seven circles in this pattern have the same size.