How many leaves on a tree? (Version 2)

Task

Amy and Greg look for a method to estimate the number of leaves on a 35 foot tree in their yard. Greg notices that the tree blocks almost all of the sunlight beneath its leaves so he thinks of the following way to estimate how many leaves are on the tree:

- We can first measure the area of the ground covered by the tree.
- Then we measure the area of an average leaf.
- We will need to estimate how much of its area an average leaf shades and how many leaves lie over an average point under the tree.

  With all of this information we should be able to get a good estimate for the number of leaves on the tree.

a. Amy and Greg find that the tree covers a region which is roughly a circle 30 feet in diameter. What is the approximate area covered by the tree?

b. How can Amy and Greg effectively measure an irregular shape such as a tree leaf?

c. How can Amy and Greg effectively decide what number to multiply by to account for multiple leaves lying over the same area and leaves shading less than their full surface area?

d. Suppose a is the approximate area underneath the tree, b the average leaf area, c the proportion of its area an average leaf shades, and d the average number of leaves found over a spot underneath the tree. In terms of the numbers a, b, c, and d what formula will Amy and Greg find for the number of leaves on the tree?

Commentary

Teachers who use this version of the task will need to bring tree leaves (or prepare a good sketch of a tree leaf) to class so that they can work on and discuss how to approximate the area of an irregular shape like a leaf. The estimated number of leaves on the tree will depend highly on the size of those leaves. So, for example, willow trees which have very small leaves can be expected to have far more leaves than a comparable sized maple tree. Note that the answer in part (b) of the question will depend on the leaves which the teacher chooses to bring to class or sketch. This is indicated in the solution below. As with ''How many leaves on a tree (version 1)'' the focus should be on the thought that goes into the model and making sure that the mathematics is then correctly executed.

In this problem, the variables a, b, c, and d are introduced to represent important quantities for this estimate: students should all understand where the formula in the solution for the number of leaves comes from. Estimating the values of these variables is much trickier and the teacher should expect and allow a wide range of variation here. The trickiest of the variables to estimate are c and d. At different parts of the tree (near the trunk versus near the perimeter for example) the number of leaves lying over a given spot may vary substantially and the students should choose a representative value and explain their choice. The type of tree under study will greatly influence how difficult these numbers are to estimate: for a maple tree, the leaves are pretty uniformly distributed throughout so this makes c and d easier to estimate.

Students who don't believe that 1,000,000 leaves on a tree is a reasonable number should consult the following: Estimating the number of leaves on a tree. The reasoning here shows some of the issues which students hopefully thought of in working through this problem.

While it seems unlikely, after working through this task and its predecessor, that this particular maple tree has 1,000,000 leaves it would not be surprising at all for a much larger tree, say in a tropical rain forest, to have well over 1,000,000 leaves.

There are many ways of trying to visualize or grasp a large number such as 1,000,000. For example, an average piece of paper is about 0.004 inches thick. So a million sheets of paper would be about

$$1{,}000{,}000 \times 0.004 = 4{,}000$$

inches thick. This is $\frac{4{,}000}{12} = \frac{1000}{3}$ feet so a little over the length of a football field. A page on a book contains about 40 lines, each with about 80 characters so this is about $40 \times 80 = 3{,}200$ characters per page. So in a 300 page book there are close to 1,000,000 characters. Of course these numbers will differ depending on the book but certainly finding a book with 1,000,000 characters will not be difficult.

Usually, in order for one quantity to be 1,000,000 times as large as another, either the smaller quantity is *very* small or the larger quantity is *very* large. The leaves on this tree fall short of 1,000,000 but the choice of a tree is a reasonable one because the leaves are large enough to easily grasp while the tree as a whole is not so colossal as to exceed what we can easily imagine.

Another good method for estimating the number of leaves on a tree would be to weigh a small number of leaves to find an estimate for the average weight of a leaf and then weigh a trash bag full of leaves to see about how many leaves fill a trash bag. Then over the course of the autumn as the leaves fall and are raked up and disposed of, an estimate for the number of bags of leaves on the tree could be found. The downside to this method is that it does not take account of the leaves that blow away and it can only be executed over a longer period of time. Moreover, the weight of the leaves will be variable depending on how much of the water inside of them has evaporated so this method requires a whole new set of variables for consideration.

Solution

a. A circle of diameter 30 feet has a radius of 15 feet and an area of

$$\pi \times 15^2 \text{ feet}^2.$$

This is a little over 700 square feet. Since the area shaded by a single leaf is likely to be measured in square inches in part (b), we will have to convert one of these two measurements. We have

$$\begin{aligned} 700 \text{ feet}^2 &= 700 \times 12^2 \frac{\text{inches}^2}{\text{feet}^2} \times \text{feet}^2 \\ &= 100{,}800 \text{ inches}^2. \end{aligned}$$

So 100,000 square inches would be a good estimate for the amount of ground shaded by this tree.

b. A good method would be to divide the leaf into a relatively small number of shapes such as rectangles and triangles for which the area can be found using algebraic formulas. As more and more smaller shapes are used to approximate the area of the leaf, the approximation will improve. Since there are many assumptions which go into the estimate for the number of leaves on a tree, it is not necessary to get an extremely accurate estimate for the area of an individual leaf.

A second good method (requiring a scale or balance) would be to make a paper cut out of the leaf. If the cutout weighs x grams and one square inch of the same paper weighs y grams then the approximate area of the leaf is $\frac{x}{y}$ grams.

Note that answers here will vary considerably depending on the kind of tree leaves being examined. Maple tree leaves could easily be 12 square inches in area or more while willow tree leaves could be a little less than 1 square inch.

c. For Greg's idea, this is likely the hardest step to perform with precision. For some trees, like weeping willows, the leaves tend to lie with relatively little of their surface area facing in the vertical direction. For a maple tree, on the other hand, the leaves lie in a variety of directions. So for this part of the estimate a larger multiple will be needed for the willow tree than for the maple (though this will be compensated for by the fact that the willow tree lets more light through than a maple tree).

The second part of this idea is also delicate, namely estimating the number of leaves which lie over a given point underneath the tree. A good way to estimate this would be to count the number of branches (of given size, preferably small) which lie over that point. Then a few branches could be studied closely to arrive at an estimate. For example, perhaps there are about ten small branches lying over a point on the surface. Studying one of these closely, it appears to shade only about half of the area underneath it. This would lead to an estimate of 5 for the number of leaves lying over that point on the ground.

To fix ideas more precisely, suppose the tree in question is a maple tree and at a few places, looking up and counting the branches overhead we come up with about 10. The leaves are relatively thick and cover about $\frac{1}{2}$ of the ground below. This would mean that we have an estimate of 5 leaves overhead for a given point. As to how much of the area of an average maple leaf will show in its shadow below the tree, suppose our estimate is $\frac{1}{2}$. This data is then used in the next part to arrive at an estimate for the number of leaves on this maple tree.

Note that the number of leaves over a given spot underneath the tree might vary considerably because the shape of the canopy of the tree is not cylindrical and so there will tend to be more leaves covering the area near the tree trunk. The best one can do is to try to estimate at several different places under the tree and make the best possible overall estimate.

d. First we go through the estimates for the data associated to the maple tree in parts (b) and (c). From part (a), the tree shades about 100,000 square inches. An average leaf is 12 square inches but only shades half of this or 6 square inches. This gives an estimate of about $\frac{100{,}000}{6}$ leaves. But this needs to be multiplied by the factor of 5 leaves lying over a given point on the ground. So we get an overall estimate of

$$\frac{5\times 100{,}000}{6}$$

leaves or a little more than 80,000.

More generally, using the variables introduced in the problem, there are about d leaves on average above the area shaded by the tree. The average area of the leaves is b and c represents the fraction of the ground shaded by an average leaf. With this data, the estimate for the number of leaves on this tree is

$$\frac{a\times d}{bc}.$$

To understand this formula, note that the total amount of ground shaded by all of the leaves is a but since there are on average d leaves over a given point this means that we could spread out the leaves to cover an area of $a\times d$. The denominator, bc, is the average area shaded by an individual leaf on the tree. Dividing the total area shaded by all leaves by the average area shaded by each leaf gives the number of leaves on the tree.

If the tree only covers a fraction, say e, of the ground below, then e would need to be added as a factor in the numerator:

$$\frac{a\times d\times e}{bc}.$$