How many leaves on a tree?

Task

Amy and Greg are raking up leaves from a large maple tree in their yard and Amy remarks "I'll bet this tree has a million leaves." Greg is skeptical and together they try to check whether or not this possible. After some thought, Amy suggests the following method:

- Let's find a very small maple tree and estimate how many leaves it has.
- Then we can use that number to figure out how many leaves are on the big maple tree.

a. Describe in detail the assumptions and calculations which need to be made in order to carry out Amy's strategy to estimate the number of leaves on the big maple tree.

b. Amy and Greg estimate that their maple tree is about 35 feet tall. Going outside, they find a 5 foot maple tree. They estimate that there about 400 leaves on the smaller tree. Use your work from part (a) to estimate how many leaves Amy and Greg would expect to find on the 35 foot maple tree.

Commentary

This is a mathematical modeling task aimed at making a reasonable estimate for something which is too large to count accurately, the number of leaves on a tree. Another method for accomplishing such an estimate will be introduced in a second task. For teachers who wish to implement these modeling tasks, a lot of flexibility is required since there are many assumptions that go into a particular model and it is not always feasible to determine how accurate they are. Nonetheless it is important that students engage in the creative use of mathematical concepts both in order to engage their imagination and to help them relate mathematics to their everyday lives. Because students will likely arrive at different estimates for the number of leaves on the tree, this would make for an ideal activity where students share with one another their reasoning. The teacher should make it clear that there is no one ''correct'' answer to this problem.

The teacher may wish to discuss the notion of similarity as it applies to this problem. The overall shape of a large maple tree and small maple tree certainly qualify as being similar. It is vital, however, that this similarity only carries over part way to the leaves. For the two maple trees in the problem, the large one is 7 times as tall as the small one (and, assuming that the shapes of the trees are similar it is also seven times as wide and deep). But the leaves of the large maple tree will not be seven times as wide, long, and thick as the leaves on the small tree. This is crucial because if the large tree really were a scaled version of the small one then it too would have 400 (very large!) leaves. So while the shape of the leaves on the large tree is similar to those of the small tree, an estimate needs to be made about their relative size which will be vital in solving the problem.

Also important in this problem is how many ''dimensions'' are considered in scaling. A tree is a three dimensional object, making three scaling factors a natural choice. But certain trees, such as some globe willows, have most of their leaves concentrated at the outermost part of the branches. The spatial area occupied by the leaves is still a three dimensional shape, in this case, but one of those dimensions (how far a leaf is from the boundary of the tree) does not necessarily scale like the others. The maple trees of the northwestern United States, which were the basis for this problem, have heavy foliage throughout and so using three scaling factors is reasonable. A weeping willow tree is another example of a tree whose foliage is pretty uniformly distributed within its canopy. This issue is particularly important if the teacher chooses to adapt the problem to a local tree whose leaves are not distributed in the same way as those on a maple tree.

This task is ideally suited for an activity which goes outside of the classroom. Students could find, in the local neighborhood or at nurseries, relatively small trees of the same species and estimate both the size and number of leaves on them. The data could then be recorded, plotted, and even statistically analyzed. Teachers implementing this task must be willing to look closely at student reasoning and expect a broad range of answers. The two keys to a complex mathematical modeling question like this are:

a. The students state clearly their hypotheses and, ideally, why these hypotheses are reasonable. Students reason clearly and accurately from their hypotheses when they calculate the number of leaves.

b. The bulk of the work, as is seen in the solution, is in the first of these two points.

Finally, we note that remarks like Amy's occur on a regular basis with no intention of examining their mathematical validity. Teachers could develop a habit of listening for claims like this from students and take advantage of good opportunities to think in greater depth about numbers and how they apply to the world around us.

Solution

a. If Amy and Greg are able to find a small maple tree, they should be able to estimate with pretty good accuracy the number of leaves on that tree. Even if counting them individually is not possible, they can use the fact that trees tend to exhibit rough radial symmetry: in other words, if they count the leaves on one half of the tree and double this it should provide a reasonable estimate for the total number of leaves on the tree. Or they might divide the tree into quarters so that counting the leaves one by one is more feasible. Since the number of leaves on the small tree is not so large, they should be able to find a relatively good estimate (very good if they are patient and do a lot of counting).

   The second part of Amy's strategy requires detailed analysis. The big maple tree is seven times as tall as the small one but this alone is not enough information to draw any conclusion about the relative number of leaves on the two trees. First, the breadth of the trees needs to taken into account. If they have the same rough shape, then this would give two more factors of seven for the other two dimensions of the tree (depth and breadth). If they do not have the same rough shape, then instead of three factors of seven we might have three different factors. For simplicity we will assume that the shapes of the trees are similar.

   There is one other subtle point to be considered which is the density of leaves in the two trees. If the larger tree is roughly seven times as high, wide, and deep as the small tree and if the density of leaves (that is, the number of leaves per fixed volume of space) is about the same then they have the information that they need to estimate the number of leaves on the larger tree. If, however, there is a large difference in this density then this would need to be factored into account. A good way to estimate this density would be to estimate the number of leaves on a branch of the large tree of comparable shape and size to the small tree.

b. We have 400 leaves on the small tree and three factors of 7 to multiply by for the height, depth, and breadth of the larger tree. This gives an estimate of

$$400 \times 7^3 = 137{,}200$$

   leaves on the large tree. If the estimated density of leaves on the large tree is, for example, one third of the density on the small tree, this gives an overall estimate of $137{,}200 \div 3 \approx 46{,}000$ leaves.

   Amy's method gives a rather large number for the estimated number of leaves on the big maple tree but not very close to the stated 1,000,000 leaves.