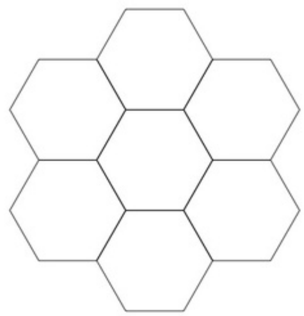
Hexagonal pattern of beehives

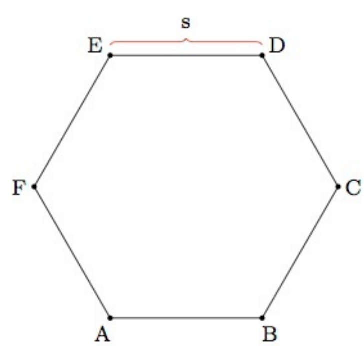
Task

Beehives are made of walls, each of the same size, enclosing small hexagonal cells where honey and pollen is stored and bees are raised. Below is a picture of some cells:



The only other regular polygons which can be used to tile the plane in this way are equilateral triangles and squares. This problem examines some of the mathematical advantages of the hexagonal tiling.

Suppose we let s denote the length of the walls in the hexagonal chambers. Below is an enlarged picture of a single hexagon:



a. Find the area of a regular hexagon H with side length s.

b. Is the ratio of area to perimeter for a regular hexagon greater than or smaller than the corresponding ratios for an equilateral triangle and a square?

c. Based on your answer to (b), why do you think it is advantageous for beehives to be built using hexagonal cells instead of triangular or square cells?

Commentary
The goal of this task is to use geometry study the structure of beehives. Beehives have a tremendous simplicity as they are constructed entirely of small, equally sized walls. In order to as useful as possible for the hive, the goal should be to create the largest possible volume using the least amount of materials. In other words, the ratio of the volume of each cell to its surface area needs to be maximized. This then reduces to maximizing the ratio of the surface area of the cell shape to its perimeter.

The hexagonal pattern of beehives is a well-documented phenomenon and many pictures can be found on-line.

Honeycomb structure

One interesting question would be why the cells in the honeycomb are not circular as a circle has a better ratio, for the same perimeter, of perimeter to area than a hexagon. One explanation for this is that unlike regular hexagons which can be put side by side to fill up space with no gaps, with circles there would be much waste. To put this another way, in the hexagonal configuration each wall is a wall for *two* separate chambers. With circles, this would still be the case but the spaces between circles would be small and oddly shaped and perhaps not very useful. Hexagons are only compared with squares and equilateral triangles as these are the only three shapes which can be put together to tile the plane (see ''Regular tessellations of the plane''). The fact that the ratio of area to perimeter for hexagons is larger than for triangles or squares shows that of these three choices, the one which provides the most space with the least amount of materials is the hexagonal pattern.

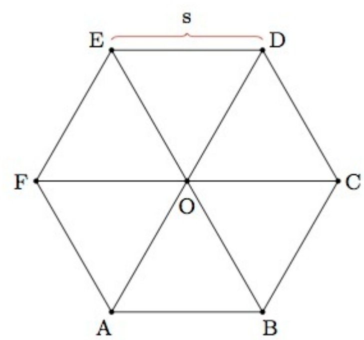
Two different solutions are presented. The first divides the hexagon into a rectangle and two triangles. For this solution, the students will need to know that the measure of the interior angles of a regular hexagon are 120 degrees. They also need to know the sine and cosine of the benchmark angles 30° and 60°. The second solution breaks the hexagon into six equilateral triangles and this requires knowing that the regular hexagon has a ''center,'' that is a point equidistant from its six vertices: this fact should be taken for granted here.

The ratio of perimeter to area for an equilateral triangle can be deduced from the results of the task ''Perimeters and areas of geometric shapes.'' The same is true of the square for which this calculation is more straightforward. Alternatively students can establish these formulas here: depending how they calculate the area of the hexagon, they will already have the tools needed to find the ratio of side length to area for the equilateral triangle.

Solutions

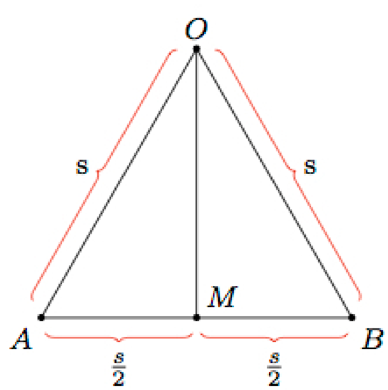
Solution: Decomposition of hexagon into triangles

a. We find the area of the hexagon by decomposing it into 6 congruent equilateral pictures as shown below:



From the picture we have that the area of the hexagon with side length s is 6 times the area of the equilateral triangle with side length s.

The area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. An argument for this calculation is provided here focusing on triangle ABO from the hexagon picture. In order to use the formula, Area $=$ $12\times$ Base $\times$ Height for a triangle, we need to choose a base and then determine the height of the triangle with this base:



Triangles AMC and BMC are congruent as we can see using using SSS:

- AM is congruent to BM since M is the midpoint of AB,
- AC is congruent to BC because triangle ABC is equilateral,
- MC is congruent to MC.

We know that $|AB| = s$. To find $|MC|$ note that since triangles AMC and BMC are congruent this means that angles AMC and BMC are congruent. Since angles AMC and BMC are supplementary, this means that they are both right angles. So we can apply the Pythagorean theorem to triangle AMC to find $|MC|$:

$$|MC|^2 + |AM|^2 = |AC|^2.$$

We know that $|AC| = s$ since AC is congruent to AB. We also know that AM is congruent to BM and, since $|AM| + |BM| = |AB| = s$, this means that $|AM| = \frac{s}{2}$. Plugging into the formula above we get

$$|MC|^2 + \frac{s^2}{4} = s^2.$$

So $|MC|^2 = \frac{3s^2}{4}$ and

$$|MC| = \frac{\sqrt{3}s}{2}.$$

Now we have found the height of triangle ABC and so the area of triangle ABC is

$$\frac{1}{2} \times s \times \frac{\sqrt{3}s}{2} = \frac{\sqrt{3}s^2}{4}.$$

The area of the hexagon is 6 times the area of the equilateral triangle and so the area of the regular hexagon is $\frac{3\sqrt{3}s^2}{2}$.

b. The ratio of area to perimeter for a regular hexagon with side length s is $\left(\frac{3\sqrt{3}s^2}{2} : 6s\right)$ which is equivalent to

$$\left(\frac{\sqrt{3}s}{4} : 1\right).$$

For an equilateral triangle, the corresponding ratio is $\left(\frac{\sqrt{3}s^2}{4} : 3s\right)$ which is equivalent

$$\left(\frac{\sqrt{3}s}{12} : 1\right).$$

So the ratio of perimeter to area for the hexagon with side length s is 3 times as large as the ratio of side length of area for the equilateral triangle with side length s. Interestingly we could make this comparison of *ratios* of area to perimeter without calculating any areas. Once the side length s is fixed the area of the hexagon with side length s is 6 times the area of the equilateral triangle with side length s. The perimeter of the hexagon is twice the perimeter of the equilateral triangle so the ratio of area to perimeter to area is three times greater for the hexagon compared to the triangle.

For a square with side length s, the perimeter is $4s$ while the area is s^2 so this gives a ratio of

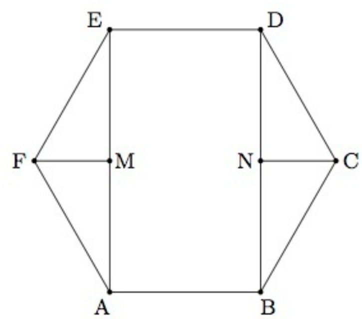
$$\left(\frac{s}{4} : 1\right)$$

for the area to the perimeter. So the area to perimeter ratio of the regular hexagon is $\sqrt{3}$ times as large as the corresponding ratio for squares.

c. There are several advantages to building a beehive in this way, tiling the plane with regular polygons. First, the walls in the construction are all simple and since the walls are all the same size the process of constructing the hive is repetitive and straightforward. Next, one big advantage to a tiling like this is that each wall serves as a wall to *two* distinct chambers, doubling the amount of useable space created with the given materials: this is the same for triangles, squares, and hexagons and does not privilege one over the others. As far as the choice between triangles, squares, and hexagons, the mathematical argument provided here shows that a hexagonal hive produces more useable space with the same amount of materials. Since it requires a lot of work to construct these hives, efficiency would dictate that the bees should make the most of their building materials.

Solution: 2 Decomposition of hexagon into a rectangle and two triangles (for part (a))

a. Here an alternative method is given for finding the area of a regular hexagon with side length s. Below is a picture of the hexagon with segments AE and BD added. Also shown are the midpoint M of segment AE and the midpoint N of BD with segments FM and CN added.



We claim that triangles AFM and EFM are congruent. Segments EM and AM are congruent because M is the midpoint of segment AE. Segments AF and EF are congruent because $ABCDEF$ is a regular hexagon. Segment FM is congruent to itself. So by SSS triangles AFM and EFM are congruent. So $m(\angle AFM) = m(\angle EFM)$. Since $m(\angle AFM) + m(\angle EFM) = m(\angle(AFE)$ and $m(\angle AFE) = 120$ we conclude that

$$m(\angle AFM) = m(\angle EFM) = 60.$$

Angles AMF and EMF are congruent and supplementary so they must both be right angles. Since the sum of the three angles in a triangle is 180 degrees we must have

$$m(\angle FEM) = m(\angle FAM) = 30.$$

We can apply the same arguments to conclude that

$$m(\angle DCN) = m(\angle BCN) = 60.$$

and

$$m(\angle CDN) = m(\angle CBN) = 30.$$

Angles FED, EDC, CBA, and BAF are all interior angles of a regular hexagon so they all are 120° angles. From this and the 30° angles calculated above, we find that

$$m(\angle BAE) = m(\angle AED) = m(\angle EDB) = m(\angle DBA) = 90$$

and so quadrilateral $ABDE$ is rectangle.

We now find the lengths of segments AM and FM and will use this to find the area of the regular hexagon. Side AF has length s by hypothesis. We have

$$\sin \angle FAM = \frac{|FM|}{|FA|}.$$

Since $m(\angle FAM) = 30$ and $\sin 30 = \frac{1}{2}$ we conclude that $|FM| = \frac{s}{2}$. Similarly

$$\cos \angle FAM = \frac{|AM|}{|FA|}.$$

Since $m(\angle FAM) = 30$ and $\cos 30 = \frac{\sqrt{3}}{2}$ we conclude that $|FM| = \frac{\sqrt{3}s}{2}$. So

$$\begin{aligned}\text{Area } (\Delta AFM) &= \frac{1}{2} \times \frac{s}{2} \times \frac{\sqrt{3}s}{2} \\ &= \frac{\sqrt{3}s^2}{8}.\end{aligned}$$

The same calculation applies to triangles EFM, DCN, and BCN. Rectangle $ABDE$ has side lengths $|AB| = s$ and $|AE| = 2 \times \frac{\sqrt{3}s}{2}$ so the area of rectangle $ABDE$ is $\sqrt{3}s^2$. Adding up the rectangle and four small triangles we find that the area of the hexagon $ABCDEF$ is $s^2 \times \sqrt{3} \times \frac{3}{2}$.