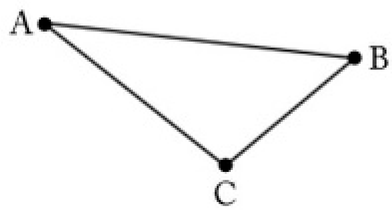
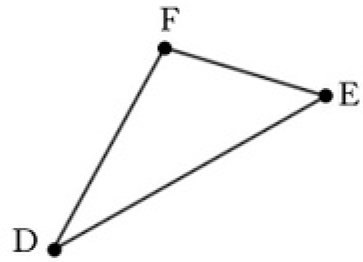
Why does SAS work?

Task

In the two triangles below, angle A is congruent to angle D, side AC is congruent to side DF and side AB is congruent to side DE:





Sally reasons as follows: "If angle A is congruent to angle D then I can move point A to point D so that side AB lies on top of side DE and side AC lies on top of side DF. Since AB and DE are congruent as are AC and DF the two triangles match up exactly and so they are congruent.''

Explain Sally's reasoning for why triangle ABC is congruent to triangle DEF using the language of reflections:

a. Construct a reflection which maps point A to point D. Call B' and C' the images of B and C respectively under this reflection.

b. Construct a reflection which does not move D but which sends B' to E. Call C'' the image of C' under this reflection.

c. Construct a reflection which does not move D or E but which sends C'' to F.

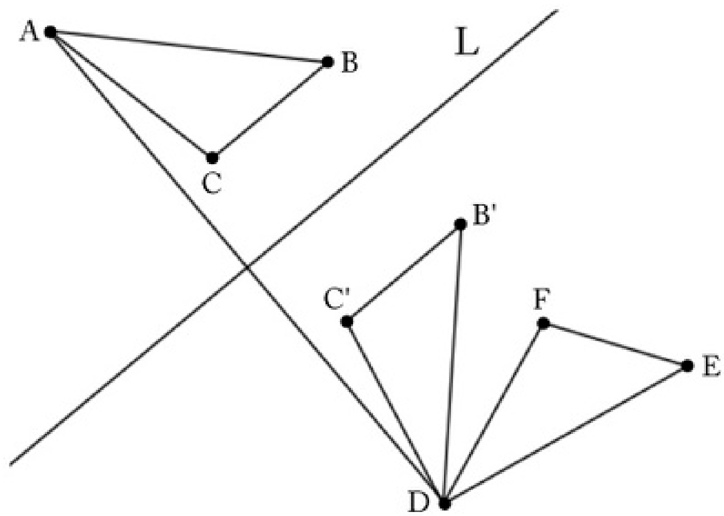
Commentary

For these particular triangles, three reflections were necessary to express how to move from ABC to DEF. Sometimes, however, one reflection or two reflections will suffice. Since any rigid motion will take triangle ABC to a congruent triangle DEF, this shows the remarkable fact that any rigid motion of the plane can be expressed as one reflection, a composition of two reflections, or a composition of three reflections.

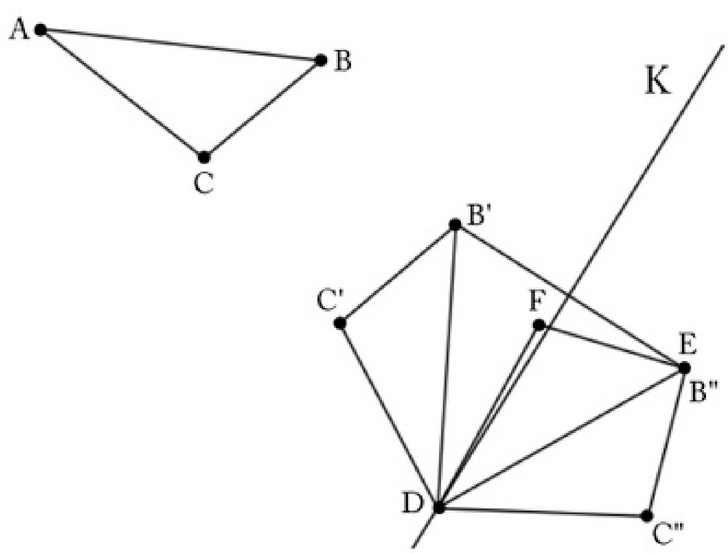
Solution

Reflection about line L sends point P in the plane to point Q exactly when L is the perpendicular bisector of PQ

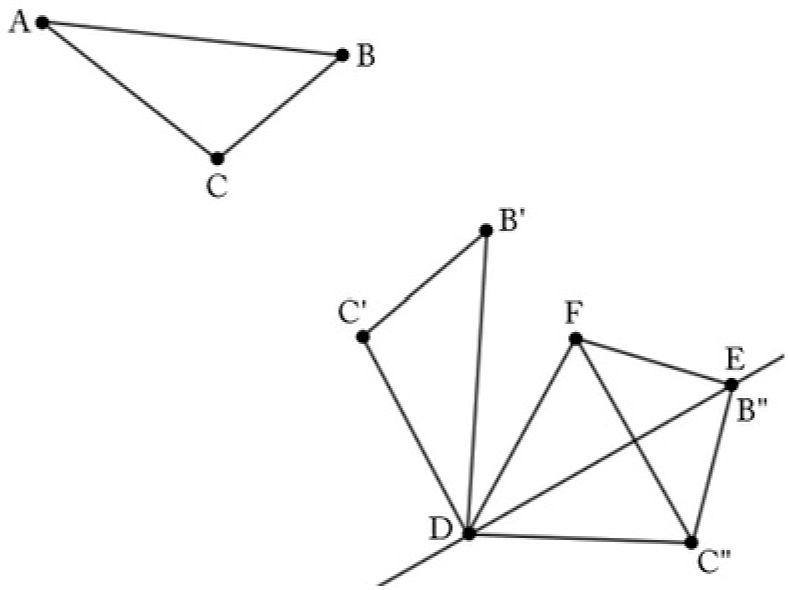
a. In the first part of this problem, we wish to send A to D via a reflection. So we must reflect about the perpendicular bisector of AD which is pictured below. Also pictured below is the new triangle $DB'C'$ obtained by reflecting triangle ABC.



b. In this step we wish to move B' to E and so we must reflect again, this time about the perpendicular bisector of $B'E$. Note that it is important that this perpendicular bisector contains D so that our second reflection preserves what we accomplished in the first step. The reason we know that D is on the perpendicular bisector of $B'E$ is that it is equidistant from E and B' by the hypothesis that AB is congruent to DE; and the perpendicular bisector of a line segment xy consists of all points in the plane equidistant from x and y. The result of the second reflection is pictured below:



c. In this last step we must move C'' to F via a reflection while leaving D and E fixed. The only reflection that leaves D and E fixed is the one about line DE. So we have to check that reflection about line DE maps C'' to F. Angle FDE is congruent to angle $C''DE$ because angle $C''DE$ is the image under two reflections of angle CAB which is congruent to angle FDE by hypothesis. Since all rigid motions of the plane preserve angles, angle $C''DE$ must map to angle FDE. Since segment DF is congruent to segment AC by hypothesis and AC is congruent to DC'' (because reflections preserve lengths of line segments) the reflection about line DE maps C'' to F. After these three reflections, the triangle ABC has been moved on top of triangle DEF so the two are congruent.



Note that the first step of this construction does not use any of the hypotheses. The second step uses the fact that AB is congruent to DE and the third step uses the facts that DF is congruent to AC and angle A is congruent to angle D.