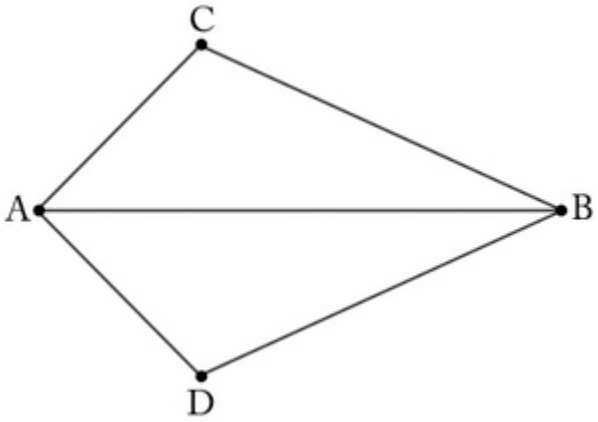
Why Does ASA Work?

Task

In triangles ABC and ABD below, we are given that angle BAC is congruent to angle BAD and angle ABC is congruent to angle ABD. Show that the reflection of the plane about line AB maps triangle ABD to triangle ABC.



Commentary
Unlike the SAS criterion for congruence, where it can be vital that the congruent angles be the ones determined by the corresponding sides, the ASA criterion for congruence works equally well for *any* two angles. This is because the sum of the angles in a triangle in the Euclidean plane is always 180 degrees. Consequently, if any two triangles ABC and DEF, not necessarily the ones in the problem, share a pair of congruent angles, say angle A is congruent to angle D and angle B is congruent to angle E, then the other pair of angles, C and F, also have to be congruent. This is because, letting m denote angle measure:

$$\begin{aligned} m(C) &= 180 - m(A) - m(B) \\ &= 180 - m(D) - m(E) \\ &= m(F) \end{aligned}$$

The two triangles in this problem share a side so that only one rigid transformation is required to exhibit the congruence between them. In general more transformations are required and the "Why does SSS work?'' and "Why does SAS work?'' problems show how this works.

Solution
We show the two triangles in the diagram are congruent by reflecting $\triangle ADB$ over the line AB and showing that the result coincides with $\triangle ACB$.

Call D' the image of D under the reflection about line AB. Since reflection about a line is a rigid motion of the Euclidean plane, it preserves both distances and angles. Since angle DAB is congruent to angle CAB by hypothesis we can conclude that angle $D'AB$ is congruent to angle CAB. Since these two angles are congruent, share ray AB and are both above line AB, we must have that ray AD' is the same as ray AC. In particular, we see that D' lies on ray AC. Reasoning in the same way with congruent angles $D'BA$ and CBA shows that D'also lies on ray BC. We know, however, that rays AC and BC meet in exactly one point, namely C, so this establishes that $D' = C$. Since reflection about line AB fixes points A and B, the triangle ABD has been mapped via reflection to triangle ABC and consequently these two triangles are congruent.