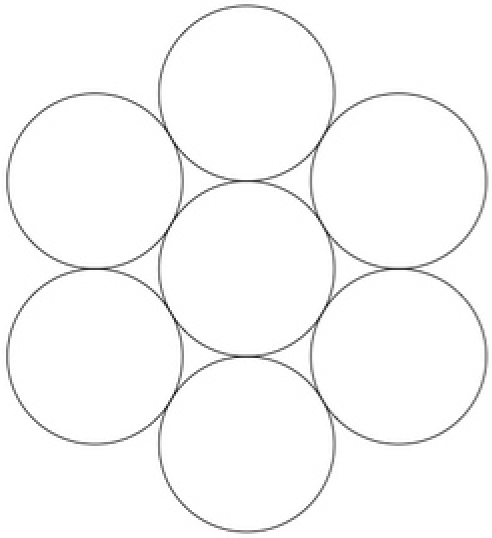
Seven Circles II

Task
Seven circles of the same size are placed in the pattern shown below:



The six outer circles touch the one in the center and each circle on the outside also touches its two neighbors in the outside ring. Find as many rigid motions of the plane as you can which are symmetries of this configuration of circles.

Commentary
This task is intended primarily for instructional purposes. It provides a concrete geometric setting in which to study rigid transformations of the plane. It is important for students to be able to visualize and execute these transformations and for this purpose it would be beneficial to have manipulatives and it will important that the students be able to label the vertices of the hexagon with which they are working. It would be helpful for the students if they have already worked through the task ''Seven Circles I'' as this will help to identify that the rotations which preserve the seven circles are multiples of 60 degrees since the regular hexagon is decomposed into six equilateral triangles: the second solution to that task focuses precisely on these rotations.

In order to provide a convincing argument that the only rigid transformations of the plane which preserve this circle configuration are the six rotations and six reflections listed, some additional knowledge about rigid transformations is required. In particular, it would be helpful to know the following fact: if A, B, and C are non-colinear points and rigid transformations of the plane T_1 and T_2 satisfy $T_1(A) = T_2(A)$, $T_1(B) = T_2(B)$, and $T_1(C) = T_2(C)$ then $T_1 = T_2$. Advanced students may want to analyze this further but it is beyond the scope of the high school standards.

Solution
One symmetry of the plane which preserves the figuration is the identity, that is the transformation which leaves every point in the plane in its original position. Setting this aside, we will look for rotations and reflections which preserve the circle configuration.

The rotations are about the center O of the circle in the middle. If we focus on a given circle in the outer ring, it can be rotated to any of the other five circles in the outer ring. Moreover, if the clockwise rotation moving the given circle to its neighbor is repeated six times, this brings us back to the original position and represents a rotation by 360 degrees. This means that each individual rotation must be one sixth of 360 degrees or 60 degrees. So the five rotations which preserve the configuration are rotations about O by $60, 120, 180, 240$, and 300 degrees. These 5 rotations can all be taken to be clockwise or counterclockwise: this is true because a rotation by 60 degrees clockwise is the same as a rotation by 300 degrees counterclockwise and similarly for the other rotations.

There are also six possible reflections: each reflection is about a line which passes through O. The lines of reflection must either pass through the centers of opposite circles or through opposite points where pairs of outer circles meet. There are only six because a given outer circle can be mapped to each of the six outer circles in one way via a reflection. The lines of reflection are pictured below: