Construction of perpendicular bisector

Task

Let A and B be two distinct points in the plane and $\overline{AB}$ the segment joining them. The goal of this problem is to construct the perpendicular bisector of segment $\overline{AB}$.

Draw circles with radius $|AB|$ centered at A and B respectively as pictured below:

The two points of intersection of these circles are labelled P and Q. Show that line $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$.

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Commentary

The construction of the perpendicular bisector of a line segment is one of the most common in plane geometry and it is undertaken here. In addition to giving students a chance to work with straightedge and compass, the problem uses triangle congruence both to show that the constructed line is perpendicular to AB and to show that it bisects AB.

This problem assumes that the two circles intersect in points A and Bas pictured. If this subject comes up, the teacher may wish to make explicit that the picture may be used as justification of this fact. Providing a more rigorous mathematical explanation goes beyond the scope of the high school standards.

A second solution is provided emphasizing rigid motions. The argument has two steps. First, it is possible to show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are perpendicular by showing that the reflection of the plane over $\overleftrightarrow{AB}$ interchanges P and Q. Once these lines are perpendicular, showing that $\overleftrightarrow{AB}$ bisects $\overline{PQ}$ can also be shown by considering a different reflection.

This task is mainly intended for instructional purposes. The first solution is a nice application of congruent triangles but is a multi-step argument. The second solution provides an opportunity for teachers to bring in the notion of rigid motions which is behind the congruence results used in the first solution. It is important for students to see results established both in the traditional way and explicitly using rigid transformations. The construction of the perpendicular bisector is a central, foundational result and so seeing this done in both ways is vital for developing a good intuition for rigid motions.

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Solutions

Solution: Congruent triangles

Notice first that triangles PAQ and PBQ pictured below are congruent.

One way to show that triangles PAQ and PBQ are congruent is with SSS. These two triangles share side $\overline{PQ}$. The sides $\overline{AQ}$, $\overline{AP}$, $\overline{BQ}$, and $\overline{BP}$ are radii of circles sharing the same radius, namely $|AB|$. So all four of these line segments are congruent and triangles PAQ and PBQ are congruent by SSS.

We now add segment $\overline{AB}$ to the above picture, labeling the point of intersection of $\overline{PQ}$ and $\overline{AB}$ as O.

We next show that triangles AQO and BQO are congruent. We know that angle OQA is congruent to angle OQB because these are the same angles as AQP and BQP which are corresponding parts of congruent triangles PAQ and PBQ. We also know that $\overline{AQ}$ is congruent to side $\overline{BQ}$ as these are also corresponding parts of congruent triangles PAQ and PBQ. Side $\overline{OQ}$ is congruent to side $\overline{OQ}$ and so, by SAS, triangle AQO is congruent to triangle BQO.

From the triangle congruence between triangles AQO and BQO we can now deduce that line $\overleftrightarrow{PQ}$ is the perpendicular bisector of segment $\overline{AB}$. Since side $\overline{AO}$ is congruent to side $\overline{BO}$ this means that line $\overleftrightarrow{PQ}$ bisects segment $\overline{AB}$. Since angle AOQ is congruent to angle BOQ and these two angles together form a line, this means that they are each 90 degree angles and so line $\overleftrightarrow{PQ}$ is perpendicular to segment $\overline{AB}$. So line $\overleftrightarrow{PQ}$ is the perpendicular bisector of segment $\overline{AB}$.

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Solution: Reflections

For this solution, we use the picture below which shows line $\overleftrightarrow{AB}$:

Let C_1 denote the circle of radius $|AB|$ with center A and C_2 the circle of radius $|AB|$ with center B. Consider the reflection of the plane, which we will denote by r, about line $\overleftrightarrow{AB}$. This line passes through the centers A and B of the two circles and so $r(A) = A$ and $r(B) = B$. Since r preserves distances, $|Ar(x)| = |AB|$ if and only if $|Ax| = |AB|$ and similarly $|r(x)B| = |AB|$ if and only if $|xB| = |AB|$. This means that $r(C_1) = C_1$ and $r(C_2) = C_2$. In particular $r(C_1)$ and $r(C_2)$ meet in the same two points, P and Q, as C_1 and C_2. Since r exchanges the points above $\overleftrightarrow{AB}$ with the points below $\overleftrightarrow{AB}$ we conclude that $r(P) = Q$ and $r(Q) = P$. By definition of reflections, we conclude that $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{PQ}$. Hence $\overline{PQ}$ is perpendicular to $\overline{AB}$ and it remains to show that it also bisects $\overline{AB}$.

Since $r(\overline{AP}) = \overline{AQ}$ and r preserves distances we know that $|AP| = |AQ|$. Since $r(\overline{BP}) = \overline{BQ}$ and r preserves distances we know that $|BP| = |BQ|$. Segment $\overline{PQ}$ is congruent to itself so by SSS we know that $\triangle\, APQ$ is congruent to $\triangle\, BPQ$. In particular $m(\angle APQ) = m(\angle BPQ)$. Let s denote the reflection of the plane about line $\overleftrightarrow{PQ}$. Since s preserves angles we conclude that $s\,(\overrightarrow{PA}) = s(\overrightarrow{PB}))$. Since s preserves distances and $|PA| = |PB|$ it follows that $s(A) = B$. The definition of reflection about $\overleftrightarrow{PQ}$ tells us that $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$ as desired.

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