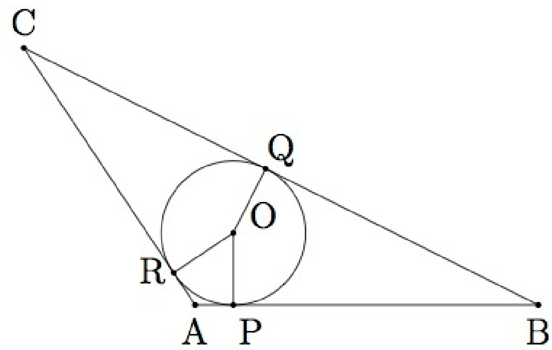
Inscribing a circle in a triangle II

Task

Suppose a circle with center O is inscribed inside triangle ABC as in the picture below:



Also pictured above are the three radii of the inscribed circle which meet the triangle at the three points P, Q, and R.

a. Show that triangle BPO is congruent to triangle BQO.

b. Show that ray $\overrightarrow{BO}$ is the bisector of angle B.

c. Show that rays $\overrightarrow{AO}$ and $\overrightarrow{CO}$ bisect the respective angles A and C.

d. Show that if a circle is inscribed in a triangle then the center of that circle is contained on all three angle bisectors for the angles of the triangle.

Commentary

This task focuses on a remarkable fact which comes out of the construction of the inscribed circle in a triangle: the angle bisectors of the three angles of triangle ABC all meet in a point. Students will need to use the fact that the radii of the inscribed circle at points P, Q, and R are perpendicular to lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$ respectively.

This task is intended mostly for instruction purposes because it relies on previous work with rigid motions of the plane and also uses the fact that the radius of a circle at a point D meets the tangent line to the circle at D perpendicularly. This task could be rewarding for students even if only approached from a construction point of view. Finding the center of the inscribed circle and seeing the three angle bisectors meeting in a point are both valuable learning experiences which might stimulate further thought. It might also be helpful to go through the steps of the problem with an equilateral triangle or a right isosceles triangle- the added symmetry of these examples would help visualize the situation and then afterward the scalene case might be easier to approach.

Solution

a. Note first that the lines containing each side of triangle ABC meet the circle in a single point and so they are tangent lines to the circle. So $\overline{OQ}$ is perpendicular to $\overleftrightarrow{BC}$ and $\overline{OP}$ is perpendicular to $\overleftrightarrow{AB}$ Hence angles BPO and BQO are congruent and they are both right angles. We also know that $\overline{OP}$ is congruent to $\overline{OQ}$ because both are radii of the same circle. Finally, $\overline{OB}$ is common to triangles BPO and BQO. Since triangles BQO and BPO are right triangles whose hypotenuses are congruent and they share one congruent leg, the Pythagorean theorem implies that the remaining pair of legs, $\overline{PB}$ and $\overline{QB}$, are also congruent. By SSS we can conclude that triangles BQO and BPO are congruent.

   Alternatively, students who know the Hypotenuse-Leg criterion for congruence can observe that triangles BQO and BPO are right triangles which share the same hypotenuse. Legs $\overline{QO}$ and $\overline{PO}$ are congruent and so the Hypotenuse-Leg criterion applies to show that triangles BQO and BPO are congruent.

b. Since triangles BPO and BQO are congruent and angles QBO and PBO are corresponding angles of these triangles they are congruent. This means that $\overrightarrow{BO}$ bisects angle B.

c. Triangles CQO and CRO are congruent by the same SSS argument used to show that triangles BPO and BQO are congruent:

   i. Side $\overline{OQ}$ is congruent to side $\overline{OR}$ since both are radii of the same circle,

   ii. Side $\overline{OQ}$ is congruent to itself.

iii. Angles CQO and CRO are right angles as shown in part (a) and so the Pythagorean theorem, combined with (i) and (ii) above, implies that the remaining pair of sides, $\overline{CR}$ and $\overline{CQ}$, are congruent.

As in part (a) above, students may also apply the Hypotenuse-Leg criterion to show that triangles CQO and CRO are congruent. These two right triangles share hypotenuse $\overline{CO}$ and have congruent legs $\overline{QO}$ and $\overline{RO}$ since these are both radii of the same circle.

Angles RCO and QCO are corresponding angles of the congruent triangles RCO and QCO so they are congruent. Hence $\overrightarrow{CO}$ bisects angle C. The same argument can be repeated a third time to show that $\overrightarrow{AO}$ bisects angle A.

d. Parts (b) and (c) show that $\overrightarrow{AO}$ is the bisector of angle A, $\overrightarrow{BO}$ is the bisector of angle B, and $\overrightarrow{CO}$ is the bisector of angle C. These angle bisectors all contain the point O at the center of the inscribed circle. The three concurrent angle bisectors are shown below: