The Circle and The Line

Task

Sketch the circle with equation $x^2 + y^2 = 1$ and the line with equation $y = 2x - 1$ on the same pair of axes.

a. There is one solution to the pair of equations

$$\begin{aligned} x^2 + y^2 &= 1 \\ 2x - 1 &= y \end{aligned}$$

that is clearly identifiable from the sketch. What is it? Verify that it is a solution.

b. Find all the solutions to this pair of equations.

Licensed by Illustrative Mathematics under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License

Solution

The equations $x^2 + y^2 = 1$ and $y = 2x - 1$ are graphed below.

The solution that is clearly identifiable from the graph, the point at which the circle and line intersect, is $(0, -1)$.

We can check that $(0, -1)$ is a solution to the pair of equations by substituting (plugging in) 0 for x and -1 for y in both equations.

$$\begin{aligned} x^2 + y^2 &= 1 \\ (0)\text{^}2 + (-1)\text{^}2 &= 1 \\ 0 + 1 &= 1 \\ 1 &= 1 \ \checkmark \end{aligned}$$

$$\begin{aligned} y &= 2x - 1 \\ -1 &= 2(0) - 1 \\ -1 &= 0 - 1 \\ -1 &= -1 \ \checkmark \end{aligned}$$

We have verified that $(0, -1)$ is a solution to the pair of equations.

Licensed by Illustrative Mathematics under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License

From the graph, we can see that there is another solution (in Quadrant I). However, it is difficult to visually determine its exact x- and y-coordinates. To find its exact location we can solve the system of equations by substitution.

Let (x, y) be the intersection point. Since $y = 2x - 1$ by virtue of the point being on the line, we can substitute the quantity $(2x - 1)$ for every y appearing in the equation of the circle. We then simplify as follows:

$$\begin{aligned}
x^2 + (2x-1)^2 &= 1 \\
x^2 + (2x-1)(2x-1) &= 1 \\
x^2 + 4x^2 - 2x - 2x + 1 &= 1 \\
5x^2 - 4x + 1 &= 1 \\
5x^2 - 4x &= 0 \\
x(5x-4) &= 0 \\
x = 0 \quad or \quad 5x - 4 &= 0 \\
5x &= 4 \\
x = \tfrac{4}{5}
\end{aligned}$$

If $x = 0$, we know $y = -1$, so we have re-discovered the first intersection point we observed. So our second intersection point has x-coordinate equal to $\frac{4}{5}$, and we are left only having to now find its y-coordinate. We simply substitute $x = \frac{4}{5}$ in either equation and solve for y.

$$y = 2x - 1 \; y \;\; = 2\left(\frac{4}{5}\right) - 1 \; y = \frac{8}{5} - 1 \; y \;\; = \frac{8}{5} - \frac{5}{5} y = \frac{3}{5}$$

Now we have that $(45, 35)$ is also a solution.

Licensed by Illustrative Mathematics under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License