Listing fractions in increasing size

Task

Order the following fractions from smallest to largest:

$$\frac{3}{8}, \quad \frac{1}{3}, \quad \frac{5}{9}, \quad \frac{2}{5}$$

Explain your reasoning.

Commentary

This is a challenging fraction comparison problem. The fractions for this task have been carefully chosen to encourage and reward different methods of comparison. The first solution judiciously uses each of the following strategies when appropriate:

- comparing to benchmark fractions,
- finding a common denominator,
- finding a common numerator.

The second and third solution shown use only either common denominators or numerators. Teachers should encourage multiple approaches to solving the problem. By working through some of the different techniques for comparing fractions, students will develop a deeper "fraction number sense." A discussion should ensue where students compare and share their methods. If all suggested methods shown in the solution given below do not come up in the discussion, the teacher could make a suggestion and let the students work further. In fact, the task is well-suited for helping students develop a strategic sense of when to use which technique.

This task is mostly intended for instructional purposes, although it has value as a formative assessment item as well. Finding the appropriate order for the fractions does not require a common denominator for all fractions, but a student who takes this legitimate approach would face computational difficulty here since the least common denominator for all four is 360, which is quite large. Students who choose a different strategy for a each fraction pair based on which approach would be easier are demonstrating a well developed fraction number sense.

Solutions

Solution: 1 Comparing with benchmark fractions and using common numerators and denominators strategically

The fractions $\frac{3}{8}$, $\frac{1}{3}$, and $\frac{2}{5}$ are all less than $\frac{1}{2}$: this can be seen, for example, by doubling the numerator and observing that twice the numerator is less than the denominator in all three cases. For $\frac{5}{9}$, on the other hand, twice 5 is bigger than 9 so if this fraction is doubled, it gives more than one and so $\frac{5}{9} > \frac{1}{2}$. This tells us which of the four fractions is largest, so now we focus on comparing the other three.
Looking at $\frac{3}{8}$, $\frac{2}{5}$, and $\frac{1}{3}$, we can see that

$$\frac{3}{9} < \frac{3}{8}$$

because 1 eighth of a whole is more than 1 ninth of the same whole, and so 3 eighths must be more than 3 ninths. But we also have

$$\frac{3}{9} = \frac{1}{3}$$

so this means that $\frac{1}{3} < \frac{3}{8}$. This is a valuable piece of information, but it does not help place $\frac{2}{5}$ relative to either $\frac{3}{8}$ or $\frac{1}{3}$. Let's compare each of these fractions individually to $\frac{2}{5}$.

There is no obvious benchmark fraction to compare $\frac{3}{8}$ and $\frac{2}{5}$. However, with only two fractions we can use a common denominator: $8 \times 5 = 40$ is a natural common denominator to use. Each $\frac{1}{8}$ needs to be subdivided into 5 equal parts to make $\frac{1}{40}$, so

$$\frac{3}{8} = \frac{3 \times 5}{8 \times 5} = \frac{15}{40}$$

Each $\frac{1}{5}$ needs to be subdivided into 8 equal parts to give $\frac{1}{40}$, and

$$\frac{2}{5} = \frac{2 \times 8}{5 \times 8} = \frac{16}{40}$$

Since

$$\frac{15}{40} < \frac{16}{40}$$

this means that

$$\frac{3}{8} < \frac{2}{5}$$

Putting this together with the work above, we have put the four fractions in order of increasing size:

$$\frac{1}{3} < \frac{3}{8} < \frac{2}{5} < \frac{5}{9}.$$

Notice that we did not need to compare $\frac{1}{3}$ and $\frac{2}{5}$ because our work shows that $\frac{3}{8}$ lies between them. If we had started by comparing $\frac{1}{3}$ to $\frac{2}{5}$ we would have found that $\frac{1}{3} < \frac{2}{5}$: this would not have told us where $\frac{3}{8}$ lies relative to $\frac{2}{5}$ since both are larger than $\frac{1}{3}$.

Solution: 2 Common Denominator

The denominators for the fractions are $8, 3, 9,$ and 5. Since 3 is a factor of 9,

$$8 \times 9 \times 5 = 360$$

will work as a common denominator. Since $8 \times (9 \times 5) = 8 \times 45$, we need to multiply the numerator and denominator by 45 to rewrite $\frac{3}{8}$ with a denominator of 360:

$$\frac{3}{8} = \frac{3 \times 45}{8 \times 45} = \frac{135}{360}.$$

For the next fraction, $\frac{1}{3}$, we have

$$\begin{aligned} 360 &= 8 \times 9 \times 5 \\ &= 8 \times 3 \times 3 \times 5 \\ &= 3 \times 8 \times 3 \times 5. \end{aligned}$$

So to get 360 as a denominator we need to multiply numerator and denominator by $8 \times 3 \times 5 = 120$:

$$\frac{1}{3} = \frac{1 \times 120}{3 \times 120} = \frac{120}{360}.$$

Proceeding in the same way with $\frac{5}{9}$ we find that the denominator 9 needs to be multiplied by $8\times5 = 40$ to get 360. Multiplying numerator and denominator by 40 gives

$$\frac{5}{9} = \frac{5\times40}{9\times40} = \frac{200}{360}.$$

Finally for $\frac{2}{5}$, to get 360 in the denominator we need to multiply by $8\times9 = 72$. Multiplying numerator and denominator by 72 gives

$$\frac{2}{5} = \frac{2\times72}{5\times72} = \frac{144}{360}.$$

Now that we have a common denominator of 360 the size of the fractions can be compared by looking at the size of the numerators. We have

$$\frac{120}{360} < \frac{135}{360} < \frac{144}{360} < \frac{200}{360}$$

because these fractions represent different numbers of equally sized parts, with 360 of these parts in a whole. Remembering the equivalences of fractions shown above, we conclude that $\frac{1}{3} < \frac{3}{8} < \frac{2}{5} < \frac{5}{9}$.

Solution: 3 Common Numerator

The numerators of these fractions, 3, 1, 5, and 2, are simpler than the denominators and so finding a common numerator is a good strategy for comparing their size. Since $1\times2\times3\times5 = 30$ we can use 30 as a common numerator. We can then rewrite each of these fractions with 30 as a numerator. This will mean multiplying numerator and denominator of each fraction by the appropriate number: 10 for $\frac{3}{8}$, 30 for $\frac{1}{3}$, 6 for $\frac{5}{9}$, and 15 for $\frac{2}{5}$:

$$\frac{3}{8} = \frac{3\times10}{8\times10} = \frac{30}{80}$$

$$\frac{1}{3} = \frac{1\times30}{3\times30} = \frac{30}{90}$$

$$\frac{5}{9} = \frac{5\times6}{9\times6} = \frac{30}{54}$$

$$\frac{2}{5} = \frac{2\times15}{5\times15} = \frac{30}{75}.$$

The more equal sized pieces a whole is broken into, the smaller those pieces will be. So, ordering the unit fractions from least to greatest, we have

$$\frac{1}{90} < \frac{1}{80} < \frac{1}{75} < \frac{1}{54}.$$

Hence we can conclude that

$$\frac{30}{90} < \frac{30}{80} < \frac{30}{75} < \frac{30}{54}$$

and so

$$\frac{1}{3} < \frac{3}{8} < \frac{2}{5} < \frac{5}{9}.$$

One variation on this theme would be to observe, as in the second solution, that $\frac{5}{9}$ is the largest of the four fractions because it is larger than $\frac{1}{2}$ while the other three are all less than $\frac{1}{2}$. This makes the rest of the ordering a little easier because now, looking at $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{8}$, we could employ the same method with a smaller common numerator, namely 6. Concretely,

$$\frac{3}{8} = \frac{3\times 2}{8\times 2} = \frac{6}{16}$$

$$\frac{1}{3} = \frac{1\times 6}{3\times 6} = \frac{6}{18}$$

$$\frac{2}{5} = \frac{2\times 3}{5\times 3} = \frac{6}{15}.$$

Reasoning as above, we see that $\frac{6}{18} < \frac{6}{16} < \frac{6}{15}$ and so once more we find that

$$\frac{1}{3} < \frac{3}{8} < \frac{2}{5} < \frac{5}{9}.$$