Using Function Notation II

Task

Given a function f, is the statement

$$f(x+h) = f(x) + f(h)$$

true for any two numbers x and h? If so, prove it. If not, find a function for which the statement is true and a function for which the statement is false.

Commentary

The purpose of the task is to explicitly identify a common error made by many students, when they make use of the "identity" $f(x+h) = f(x) + f(h)$. A function f cannot in general be distributed over a sum of inputs. This is an easy mistake to make because

$$f(x+h) = f(x) + f(h)$$

is a true statement if f, x, h are real numbers and the operations implied by the parentheses are multiplication. The task has students find a single explicit example for which the identity is false, but it is worth emphasizing that in fact the identity fails for the vast majority of functions. Among continuous functions, the *only* functions satisfying the identity for all x and h are the functions $f(x) = ax$ for a constant a.

Solution

The statement is not true for all functions.

A function for which it holds is the function f given by $f(a) = 5a$. If $f(a) = 5a$, then

$$f(x+h) = 5(x+h) = 5 \cdot x + 5 \cdot h = f(x) + f(h).$$

A function for which the statement does not hold is the function f given by $f(a) = a^2$. If $f(a) = a^2$, then

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2.$$

This differs from

$$f(x) + f(h) = x^2 + h^2$$

by $2xh$. This middle term is not zero unless x or h is zero.