Parabolas and Inverse Functions

Task

a. Explain why the equation $y = x^2$ represents y as a function of a real variable x.

b. For the relation considered in part (a), is x a function of y? Explain.

c. Give a context in which the equation $y = x^2$ does represent x as a function of y.

Commentary

This problem is a simple de-contextualized version of "Your Father" and "Parking Lot". It also provides a natural context where the absolute value function arises as, in part (b), solving for x in terms of y means taking the square root of x^2 which is $|x|$.

This task assumes students have an understanding of the relationship between functions and equations. Using this knowledge, the students are prompted to try to solve equations in order to find the inverse of a function given in equation form: when no such solution is possible, this means that the function does not have an inverse. Part (c) is an open-ended question which, with teacher guidance, leads students to realize that this problem can be fixed by restricting the domain of the function to a smaller set.

Solution

a. The equation $y = x^2$ makes y a function of x because for each real number x_0, the equation $y = x^2$ assigns y a value of x_0^2. So the equation $y = x^2$ naturally gives rise to the function f defined by the rule $f(x) = x^2$.

b. For most values of y, such as $y = 1$, there are two values of x, $x = 1$ and $x = -1$ in this case, which satisfy $y = x^2$. If we could write x as a function of y then each y value would correspond to only one x value. So x cannot be written as a function of y if we try to do this for all real numbers x. The relation in part (a) is for all real numbers x and so x is not a function of y in this setting.

c. Looking at part (b), x can be viewed as a function of y as long as we do not include pairs of values of equal distance from 0, such as $x = +1$ and $x = -1$, in the set of x values. For example, if x is restricted to the non-negative real numbers, we have the equation $x = \sqrt{y}$ describing x as a function of y. Another way to express this is the following: when x is restricted to the non-negative real numbers, each non-negative y-value corresponds to exactly one non-negative x value so x can be viewed as a function of y. Similarly, if x is restricted to the non-positive real numbers, we can represent x as a function of y by the equation $x = -\sqrt{y}$.

   In fact, the set of x-values considered could be quite complex -- it need only avoid containing both the values a and $-a$ for a non-zero real number a. For example, the set of x-values could be all real numbers greater than 3 together with those (negative) real numbers between -2 and -1. The set could also be extremely simple: For example, if only the value 5 is allowed for x then the equation $y = x^2$ makes x a function of y.