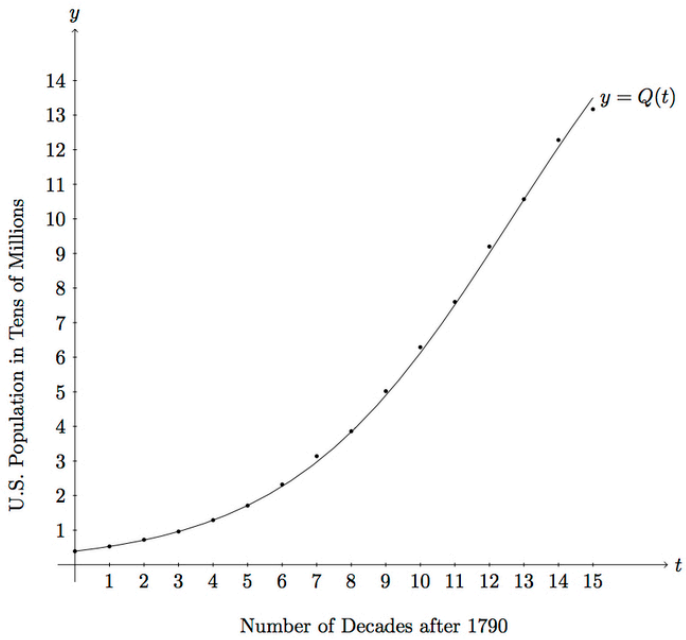
Logistic Growth Model, Explicit Version

Task

The dots in the graph below show the approximate United States Population measured each decade starting in 1790 up through 1940:



The curve above, modeling the United States population, is the graph of the function P given by the rule

$$P(t) = \frac{(3{,}900{,}000 \times 200{,}000{,}000)e^{0.31t}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31t} - 1)}.$$

a. According to this model for the U.S. population, what was the population in the year 1790?
b. According to this model for the U.S. population, when did the population first reach 100,000,000? Explain. How much does this differ from an estimate that the U.S. population first reached 100,000,000 in 1915?
c. According to this model, what should the population of the U.S. be in the year 2010? How much does this differ from the Wikipedia estimate of 309,000,000?
d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Why?

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Commentary

This problem introduces a logistic growth model in the concrete setting of estimating the population of the U.S. The model gives a surprisingly accurate estimate and this should be contrasted with linear and exponential models, studied in "U.S. Population $1790 - 1860$." Indeed, a linear model predicts constant differences in the U.S. population over each decade and this is far from the actual data. An exponential model would predict constant quotients in successive populations over each decade and this is also not a good fit to this data (though it is better than the linear model). Finally, a quadratic model would predict equal second differences in the U.S. population whereas the actual data show a large variation in second differences over this period of 150 years. The data does not fit any of these models (logistic, exponential, linear, or quadratic) perfectly. If time permits, it would be an interesting project to study all four models comparing strengths and weaknesses of each. While quantifying the strength of the different models is beyond the scope of the high school standards, the question could be raised for students with a strong desire to explore further.

The expression for the modeling function P looks very complicated but in fact, as working through the exercise aims to show, the model is relatively simple. The number $3{,}900{,}000$ is the population in 1790 when the model starts. The number $200{,}000{,}000$ is the number that this model predicts that the U.S. population will approach as time goes on. Finally the number 0.31 determines how quickly the population approaches the limiting value of $200{,}000{,}000$.

Students should be encouraged to experiment with changing the value 0.31 and also the value $200{,}000{,}000$ and see what the impact is on the shape of the graph of P. The goal of this task is both to give the student some experience working with exponential functions while also introducing this very important model which has a surprising flexibility considering that there are only three numbers to choose in the formula for P. Due to the awkward numbers, this task is probably best suited for instruction purposes.

Note that the appropriate language for part (d) is that of limits, taking the limit as t approaches infinity. For students who are familiar with this notation and language, the teacher should call this to their attention.

The data points used in the plot were taken from Wikipedia. The estimate that the U.S. population first reached $100{,}000{,}000$ in 1915 came from Facts and Figures.

Solution

a. Since $P(t)$ is the U.S. population t years after 1790, the population in 1790 is given by $P(0)$. Looking at $P(0)$, the only place where the variable t appears in the expression defining P is in the exponentials $e^{0.31t}$ which are in both the numerator and denominator. Since $e^0 = 1$, this means that

$$P(0) = \frac{3{,}900{,}000 \times 200{,}000{,}000}{200{,}000{,}000 - 0} = 3{,}900{,}000.$$

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So the approximate U.S. population in 1790 is the term $3{,}900{,}000$ appearing in the numerator of $P(t)$.

b. Setting $P(t) = 100{,}000{,}000$ we have

$$\frac{3{,}900{,}000 \times 200{,}000{,}000 \times e^{0.31t}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31t} - 1)} = 100{,}000{,}000.$$

All of the zeroes here create a distraction, making the work more difficult. Dividing the numerator and denominator on the left by $1{,}000{,}000$ and dividing both sides of the equation by $100{,}000{,}000$ simplifies this equation to

$$\frac{3.9 \times 2 \times e^{0.31t}}{200 + 3.9(e^{0.31t} - 1)} = 1.$$

Clearing the denominator on the left gives

$$3.9 \times 2 \times e^{0.31t} = 200 + 3.9(e^{0.31t} - 1).$$

Moving the exponential terms to the left hand side of the equation and the numbers to the right hand side, we find, after simplifying:

$$3.9 \times e^{0.31t} = 196.1$$

Solving for t gives a value of about $t = 12.64$. This means that it would be about 126.4 years after 1790 that the population reaches 100 million. This would be during the year 1916. This differs by one year from the actual data, a little less than one percent of the 125 years it took for the population to reach $100{,}000{,}000$.

c. Since 2010 is 220 years after 1790, or 22 decades, the model predicts that in 2010 the population of the U.S. should have been

$$P(22) = \frac{3{,}900{,}000 \times 200{,}000{,}000 \times e^{0.31\times 22}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31\times 22} - 1)}.$$

Evaluating on a calculator gives an estimate of between $189{,}000{,}000$ and $190{,}000{,}000$. This estimate is too low, differing by about $119{,}000{,}000$ from the estimated U.S. population of $309{,}000{,}000$ in 2010.

There has been a large growth in the population of the United States since 1940. Many factors have influenced this growth, including the general prosperity and availability of resources that has been in place since that time and also the large number of immigrants who have come to the United States from all over the world.

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d. Note that $P(50)$ would represent the estimated U.S. population 50 decades after 1790 or, in other words, in the year 2290. Evaluating on a calculator gives a population, rounded to the nearest one, of $199{,}998{,}134$. Plugging in a larger value, such as $t = 100$ gives a value of $200{,}000{,}000$, again to the nearest one. The estimate of $200{,}000{,}000$ remains in place for all values of t larger than 100 as well.

To see why, in terms of the structure of the expression defining P, the model predicts a population approaching and stabilizing at $200{,}000{,}000$, examine first the denominator of P:

$$200{,}000{,}000 + 3{,}900{,}000e^{0.31t} - 3{,}900{,}000.$$

The exponential term $3{,}900{,}000e^{0.31t}$ grows rapidly as t grows. The rest of the denominator, $200{,}000{,}000 - 3{,}900{,}000$, does not depend on t. So as t grows, the denominator is better and better approximated by $3{,}900{,}000\mathrm{e}^{0.31\mathrm{t}}$. The numerator is $3{,}900{,}000 \times 200{,}000{,}000e^{0.31t}$. Taking the quotient of $3{,}900{,}000 \times 200{,}000{,}000e^{0.31t}$ by, $3{,}900{,}000e^{0.31t}$ the approximation of the denominator when t is large, gives $200{,}000{,}000$. So as t grows, the values of P become closer and closer to $200{,}000{,}000$.

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