Combined Fuel Efficiency

Task

The US Department of Energy keeps track of fuel efficiency for all vehicles sold in the United States. Each car has two fuel economy numbers, one measuring efficient for city driving and one for highway driving. For example, a 2012 Volkswagen Jetta gets 29.0 miles per gallon (mpg) in the city and 39.0 mpg on the highway.

Many banks have “green car loans” where the interest rate is lowered for loans on cars with high combined fuel economy. This number is not the average of the city and highway economy values. Rather, the combined fuel economy (as defined by the federal Corporate Average Fuel Economy standard) for x mpg in the city and y mpg on the highway, is computed as

$$\text{combined fuel economy} = \frac{1}{\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)}.$$

a. What is the combined fuel economy for the 2012 Volkswagen Jetta? Give your answer to three significant digits.

b. For most conventional cars, the highway fuel economy is 10 mpg higher than the city fuel economy. If we set the city fuel economy to be x mpg for such a car, what is the combined fuel economy in terms of x? Write your answer as a single rational function $\frac{a(x)}{b(x)}$.

c. Rewrite your answer from (b) in the form of $q(x)+\frac{r(x)}{b(x)}$ where and $q(x)$, $r(x)$ and $b(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $b(x)$.

d. Use your answer in (c) to conclude that if the city fuel economy, x, is large, then the combined fuel economy is approximately $x+5$.

Commentary

The primary purpose of this problem is to rewrite simple rational expressions in different forms to exhibit different aspects of the expression, in the context of a relevant real-world context (the fuel efficiency of a car). Indeed, the given form of the combined fuel economy computation is useful for direct calculation, but if asked for an approximation, is not particularly helpful. However, in a reduced form, $\frac{q(x)+r(x)}{b(x)}$, it is easy to read off the approximate value as $q(x)$ when $\frac{r(X)}{b(x)}$ is small.

In part (a), the Corporate Average Fuel Economy (CAFE) law passed for cars made between 2012 and 2016 requires that computations be made to five significant figures before rounding. Most car manufacturers report CAFE values to three significant figures, and students are asked to do the same in the task. In part (d), students can explore how large of an x value is needed for the approximation to be, for example, within 1 mpg of the actual combined fuel economy.

Part (c) might be approached in the context either of instruction or of exploration. Since no specific method is requested or offered for the reduction to $\frac{q(x)+r(x)}{b(x)}$, the teacher may want to review or introduce a specific method as part of the problem.

Some notes on the combined fuel economy: Though the task does not mention the term, the CAFE standard defines the combined fuel economy as the harmonic mean of the city and highway fuel economy, as defined in the Federal Code, Title 40 Chapter 1 Subsection Q Part 600 Subpart C Section 600.206-08 a.3. This is because the Department of Energy measures efficiency not in terms of miles per gallon but in terms of the reciprocal unit of gallons per mile. The arithmetic mean of the city and highway measurements in gallons per mile corresponds precisely to the harmonic mean of the city and highway measurements in miles per gallon, giving the formula in the problem statement. It is interesting to note that the answer $x+5$ of in part (d) reflects that the harmonic mean of x and $x+10$ approximates the arithmetic mean of those two quantities for large values of x.

For reference, the text of the federal statute is as follows:

> “For the purpose of determining average fuel economy under §600.510-08, the combined fuel economy value for a vehicle configuration is calculated by harmonically averaging the FTP-based city and HFET-based highway fuel economy values"

In the statute they go on to give slightly more weight to the city than highway when computing the harmonic mean. A sample calculation is given in Federal Code, Title 40 Chapter 1 Subsection Q Part 600 Appendix II a.2.

Current web addresses for the text:

Cornell Law
Cornell Law

Solution

At least by part (b), it becomes more convenient to re-write the expressions by

$$\frac{1}{\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)} = \frac{2xy}{x+y}.$$

We'll do this at the start, though part (a) could certainly be done using the original form of the expression.

a. For $x = 29.0$ mpg and $y = 39.0$ mpg, compute that

$$\text{combined fuel economy} = \frac{2(29.0)(39.0)}{29.0+39.0} \approx 33.265\ .$$

To three significant digits, this is 33.3 mpg.

We note that this exercise is an opportunity to pay close attention to units, especially since the units of a harmonic mean of two quantities are not immediately obvious. Represent mpg as $\frac{\text{miles}}{\text{gallon}}$ the same computation done with units looks like

$$\frac{2\left(29.0\frac{\text{miles}}{\text{gallon}}\right)\left(39.0\frac{\text{miles}}{\text{gallon}}\right)}{29.0\frac{\text{miles}}{\text{gallon}}+39.0\frac{\text{miles}}{\text{gallon}}} \approx 33.265\frac{\left(\frac{\text{miles}}{\text{gallon}}\right)^2}{\frac{\text{miles}}{\text{gallon}}} = 33.265\,\frac{\text{miles}}{\text{gallon}}.$$

b. For $y = x,$ we have

$$\text{combined fuel economy} = \frac{2x(x+10)}{2x+10} = \frac{2x(x+10)}{2(x+5)} = \frac{x(x+10)}{(x+5)}.$$

c. A student might calculate this reduction using long division, synthetic division or grouping. For any method, we have

$$\frac{x(x+10)}{x+5} = x + 5 - \frac{25}{x+5}$$

d. When x is large, $\frac{25}{(X+5)}$ is small. In particular, when $x > 20,$ this term is less than 1 so the approximation of $x + 5$ is within 1 mpg of the correct value of the combined fuel economy.