Building a General Quadratic Function

Task

In "Building an explicit quadratic function" a particular quadratic function was rewritten by completing the square. The quadratic function used was $q(x) = 2x^2 + 4x - 16$ and this function was rewritten as

$$q(x) = 2(x+1)^2 - 18 = 2((x+1)^2 - 9).$$

Some of the advantages to this form are that the x-coordinate of the vertex of the graph of q can be found more easily and the two roots of q can also be found readily. The right hand side of this equation can be seen as a horizontal translation by -1, then squaring, then a vertical translation by -9, and finally a multiplicative scaling by 2. The goal of this task is first to employ the same technique on a general quadratic function and then derive the quadratic formula. To assist in this process, we first rewrite the equation above:

$$\frac{q(x)}{2} = (x+1)^2 - 9.$$

Let f be a quadratic function, so we have

$$f(x) = ax^2 + bx + c.$$

Here $a,\ b,\ c$ are real numbers and we assume that a is non-zero.

a. Following the lead of our example problem, we begin by dividing out the leading coefficient:

   Multiplying $f(x)$ by $\frac{1}{a}$ gives

   $$\frac{f(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}.$$

   We wish to write this as the square of a linear function $x + d$ plus a constant. Find numbers d and k so that $\frac{f(x)}{a} = (x+d)^2 + k$.

b. Using the values for d and k found in part (b), rewrite $f(x) = ax^2 + bx + c$ as

   $$\frac{f(x)}{a} = (x+d)^2 + k$$

   Why do the expressions $f(x)$ and $\frac{f(x)}{a}$ have the same roots?

c. Explain how to deduce the quadratic formula for the roots of f from part (c).

Commentary

This task is for instructional purposes only and builds on "Building an explicit quadratic function."

First, it is vital that students have worked through "Building an explicit quadratic function" before undertaking this task. The instructor may wish to recall the steps used to manipulate the quadratic function $q(x) = 2x^2 + 4x - 16$ given in the following series of equations:

$$q(x) = 2x^2 + 4x - 16$$
$$\frac{q(x)}{2} = x^2 + 2x - 8$$
$$\frac{q(x)}{2} = (x+1)^2 - 1 - 8$$
$$\frac{q(x)}{2} = (x+1)^2 - 9$$

The last equation can also be written as

$$q(x) = 2((x+1)^2 - 9)$$

In this form, the function q is seen to arise from 4 different operations:

a. a horizontal shift by -1,
b. squaring,
c. a vertical shift by -9,
d. a multiplicative scaling by 2.

This task performs the same operations with a general quadratic function $f(x) = ax^2 + bx + c$. The operations take place in the opposite order because f is the starting place and the goal is to slowly decode f to find the horizontal shift, the vertical shift, and the multiplicative scaling as was done for q. The first step is to find the multiplicative scaling and the other three steps, the horizontal and vertical displacements and the squaring operation are done together via the process known as completing the square, corresponding to the third displayed equation above in the case of $q(x) = 2x^2 + 4x - 16$.

Working with so many variables, as are present in this problem, can be a challenge. An alternative or a warm-up to this problem would be to work with several concrete quadratic functions, like $q(x) = 2x^2 + 4x = 16$, and run through this problem to find, in part (c), the expression $2((x-1)^2 - 9)$ for $q(x)$ which then allows to find the roots of $q(x)$ as was done in "Building an explicit quadratic function." Students who can manipulate these expressions with explicit numbers rather than variables are on very solid footing with respect to the standard F-BF.3.

The quadratic formula, which arises naturally in part (d), can also be derived in a purely geometric way which explains the name "completing the square" often given to part (b) of this process. In the method used here, there is also an underlying geometric intuition but it has to do with the graphs of the functions at the different steps of the process.

Solution

a. Note first that we can only multiply the quadratic polynomial $f(x)$ by $\frac{1}{a}$ if a is non-zero: this is the one place in this problem where the hypothesis $a \neq 0$ is used.

If we square the linear function $x + d$ we find

$$(x + d)^2 = x^2 + 2dx + d^2$$

Here we wish to write

$$x^2 + \frac{b}{a}x + \frac{c}{a}$$

as $(x + d)^2$ for numbers d and k. Since the coefficient of x in $(x + d)^2$ is $2d$ and the coefficient of x in $x^2 + \frac{b}{a}x + \frac{c}{a}$ is $\frac{b}{a}$ this means that we want to choose d to be

$$d = \frac{b}{2a}.$$

Now that we have found d, we see that

$$\begin{aligned}(x + d)^2 &= \left(x + \frac{b}{2a}\right)^2 \\ &= x^2 + \frac{b}{a}x + \frac{b^2}{4a} \\ &= x^2 + \frac{b}{a}x + \frac{c}{a} + \left(\frac{b^2}{4a} - \frac{c}{a}\right).\end{aligned}$$

This tells us that

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x + d)^2 - \left(\frac{b^2}{4a} - \frac{c}{a}\right)$$

then the constant in the expression $x^2 + \frac{b}{a}x + \frac{c}{a} = (x + d)^2 + k$

$$k = -\left(\frac{b^2}{4a} - \frac{c}{a}\right).$$

b. In part (a) we found that $d = \frac{b}{2a}$ and $k = -\left(\frac{b^2}{4a} - \frac{c}{a}\right)$. Plugging these into the expression $\frac{f(x)}{a} = (x + d)^2 + k$ gives

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right).$$

To see why $\frac{f(x)}{a}$ and $f(x)$ have the same roots, suppose first that r is a root of $\frac{f(x)}{a}$ so that $\frac{f(r)}{a} = 0$. This means that $f(r) = 0$ and so r must be a root of f. On the other hand, if $f(r) = 0$ then

$$\frac{f(r)}{a} = \frac{0}{a} = 0$$

and so r is a root of $\frac{f(x)}{a}$.

c. From part (b) we have

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right).$$

Since the roots of $f(x)$ are the same as the roots of $\frac{f(x)}{a}$, also from part (c), we can find the roots of f by setting the right hand side of this equation equal to zero, obtaining

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

Taking the square root of both sides gives

$$\left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

This means that

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

Moving $\frac{b}{2a}$ to the other side for the equation gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is the quadratic formula.