Zeroes and factorization of a quadratic polynomial II

Task

Suppose f is a quadratic function given by the equation $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and a is non-zero.

a. Explain why f can have at most two roots; that is explain why there can be at most two distinct real numbers r_1, r_2 so that $f(r_1) = f(r_2) = 0$.

b. Give examples to show that it is possible for f to have zero, one, or two real roots.

Commentary

This task continues “Zeroes and factorization of a quadratic polynomial I.” The argument here generalizes, as shown in “Zeroes and factorization of a general polynomial” to show that a polynomial of degree d can have at most d roots. In the quadratic case, an alternative argument for why there can be at most two roots can be given using the quadratic formula and this is done in the second solution below.

This task is intended for instructional purposes to help students see more clearly the link between factorization of polynomials and zeroes of polynomial functions. Students who are familiar with the quadratic formula should be encouraged to think about the first solution which extends to polynomials of higher degree where formulas for the roots are either very complex or not possible to find.

Solutions

Solution: 1 Algebra

a. The first important result which helps to analyze the roots of f is given in "Zeroes and factorization of a quadratic polynomial I." There it was shown that if $f(r) = 0$ then $f(x)$ is evenly divisible by $x - r$. Applying this here, suppose r_1 and r_2 are distinct real numbers and $f(r_1) = f(r_2) = 0$. Since $f(r_1) = 0$ we know that $(x - r_1)$ evenly divides $ax^2 + bx + c$. This means that we can write

$$ax^2 + bx + c = (x - r_1)l(x)$$

where $l(x)$ is a linear function of x.

Since r_2 is also a root of f, we have that $f(r_2) = 0$ and so

$$\begin{aligned} f(r_2) &= ar_2^2 + br_2 + c \\ &= (r_2 - r_1)l(r_2) \\ &= 0 \end{aligned}$$

Now $r_2 \neq r_1$by assumption so $r_2 - r_1 \neq 0$. Since the product $(r_2 - r_1)l(r_2)$ is zero, this means that $l(r_2) = 0$. Since the coefficient of x in $l(x)$ is a this means that $l(x) = a(x - r_2)$. So we can write

$$f(x) = a(x - r_1)(x - r_2).$$

Now if r_3 were a third root of f, distinct from r_1 and r_2, we would have

$$0 = f(r_3) = a(r_3 - r_1)(r_3 - r_2).$$

This is not possible, however, because $a \neq 0$, $r_3 - r_1 \neq 0$, and $r_3 - r_2 \neq 0$.

b. There are many possible answers to this question. A simple example of a quadratic polynomial with no real zeroes is $x^2 + 1$ which has roots $\pm i$ where i represents$\sqrt{-1}$. An example of a polynomial with one real root is x^2 which has only 0 as a root. And an example of a polynomial with two real roots is $x^2 - 1$, which has roots ± 1. In general, as is shown in the second solution below, the quadratic formula says that for $f(x) = ax^2 + bx + c$, the number of real roots is determined by the expression

$$b^2 - 4ac.$$

When this is negative, there are no real roots. When it is zero there is one real root and when it is positive there are two real roots.

Solution: 2 Quadratic Formula

According to the quadratic formula (see "Building a general quadratic function") the two roots to $ax^2 + bx + c$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This gives us a lot of information concerning the roots of f. First it tells us that there can be at most two real numbers r_1, r_2 with $f(r_1) = f(r_2) = 0$, namely the values of x listed above. The reason there are not always two real roots is that the radical expression $\sqrt{b^2 - 4ac}$ could be imaginary, in which case there are no real roots of f. This corresponds to the case where $b^2 - 4ac < 0$. Next, we could have $b^2 - 4ac = 0$ and in this case there is only one real root, which is repeated twice. Finally it could be that $b^2 - 4ac > 0$ and then there are exactly two real roots of f. To give explicit examples, it suffices to choose whole number $a,\ b,\ c$ with $b^2 - 4ac$ being negative, zero, or positive. The hardest case to produce is the one where f has only one real root. This is not surprising because if we think of the graph of f, this means that the vertex is on the x-axis.