Compounding with a 5% Interest Rate

Task

A man invests \$1000 in an account with a 5% annual interest rate. He knows that money in an account where interest is compounded semi-annually will earn interest faster than money in an account where interest is compounded annually. He wonders how much interest can be earned by compounding it more and more often. In this problem we investigate his question.

If the man's interest is compounded annually, his year-end balance will be:

$$\begin{aligned} \$1000 + 5\% \times \$1000 &= \$1000 + 0.05 \times \$1000 \\ &= \$1000(1 + 0.05) \\ &= \$1050 \end{aligned}$$

If his interest is compounded semi-annually, he earns half the annual interest at mid-year, and his mid-year balance is:

$$\begin{aligned} \$1000 + \frac{5\%}{2} \times \$1000 &= \$1000 + \frac{0.05}{2} \times \$1000 \\ &= \$1000\left(1 + \frac{0.05}{2}\right) \\ &= \$1025 \end{aligned}$$

At year-end he earns the other half of his annual interest and his year-end balance is:

$$\begin{aligned} \$1025 + \frac{5\%}{2} \times \$1025 &= \$1025 + \frac{0.05}{2} \times \$1025 \\ &= \$1025\left(1 + \frac{0.05}{2}\right) \\ &= \$1000\left(1 + \frac{0.05}{2}\right)\left(1 + \frac{0.05}{2}\right) \\ &= \$1000\left(1 + \frac{0.05}{2}\right)^2 \\ &= \$1050.625 \end{aligned}$$

a. Find the end of year balance if the interested is compounded quarterly.

b. Write an expression which gives the man's end of year balance in terms of the number of times the interest is compounded, n.

c. Substitute $k = \frac{0.05}{n}$ into your expression so that the whole expression is written in terms of k instead of in terms of n.

d. Now we'll investigate what happens to the end of year balance as we compound the interest more and more. This means that we want to increase the value of n. What does increasing the value of n do to the value of k?

e. Complete the table below to help you see what happens to the end of year balance as k becomes larger and larger. Round to the 5th decimal place.

k	$(1+k)^{\frac{1}{k}}$
0.1	
0.01	
0.001	
0.0001	
0.00001	

The values in the second column of your table should not appear to be growing out of control. They should appear to approach a limiting value. This value is an irrational number which mathematicians denote with the letter e.

f. Based on the results of your table, what value does it appear the end of year balance will approach as the interest is compounded more and more often? Write this value in terms of e.

Commentary

In part a), students should use a calculator, such as a Texas Instruments 84, which allows them to keep exact answers for use in subsequent calculations.

Note that banks report account balances to the nearest cent. However, in this task we have kept several decimal places of accuracy in the account balance to clearly show changes in the account balance beyond the second decimal place.

Solution

a. At the end of March, the man will earn one quarter of the interest:

$$\begin{aligned}\$1000 + \frac{5\%}{4}\times\$1000 &= \$1000 + \frac{0.05}{4}\times\$1000\\ &= \$1000\left(1+\frac{0.05}{4}\right)\\ &= \$1012.5\end{aligned}$$

At the end of June, the man will earn the next quarter of the interest:

$$\begin{aligned}\$1012.5 + \frac{5\%}{4}\times\$1012.5 &= \$1012.5 + \frac{0.05}{4}\times\$1012.5\\ &= \$1012.5\left(1+\frac{0.05}{4}\right)\\ &= \$1000\left(1+\frac{0.05}{4}\right)\left(1+\frac{0.05}{4}\right)\\ &= \$1000\left(1+\frac{0.05}{4}\right)^2\\ &= \$1025.15625\end{aligned}$$

At the end of September, the man will earn the next quarter of the interest:

$$\begin{aligned}\$1025.15625 + \frac{5\%}{4}\times\$1025.15625 &= \$1025.15625 + \frac{0.05}{4}\times\$1025.15625\\ &= \$1025.15625\left(1+\frac{0.05}{4}\right)\\ &= \$1000\left(1+\frac{0.05}{4}\right)^2\left(1+\frac{0.05}{4}\right)\\ &= \$1000\left(1+\frac{0.05}{4}\right)^3\\ &= \$1037.970703125\end{aligned}$$

At the end of December, the man will earn the next quarter of the interest:

$$
\begin{aligned}
\$1037.970703125+\frac{5\%}{4}\times\$1037.970703125 &= \$1037.970703125+\frac{0.05}{4}\times\$1037.970703125\\
&= \$1037.970703125\left(1+\frac{0.05}{4}\right)\\
&= \$1000\left(1+\frac{0.05}{4}\right)^3\left(1+\frac{0.05}{4}\right)\\
&= \$1000\left(1+\frac{0.05}{4}\right)^4\\
&\approx \$1050.95
\end{aligned}
$$

b. Organizing and generalizing our previous work, we have, putting the number of times the interest is compounded in the first column and the expression showing the year end balance in the second column:

1	$\$1000(1+\frac{0.05}{1})^1$
2	$\$1000(1+\frac{0.05}{2})^2$
4	$\$1000(1+\frac{0.05}{4})^4$
n	$\$1000(1+\frac{0.05}{n})^n$

c. Substituting $k=\frac{0.05}{n}$ yields $\$1000(1+k)\frac{0.05}{k}$.

d. Increasing the value of n decreases the value of k, since increasing the value of the denominator of a fraction while holding the numerator constant decreases the value of the fraction.

e.

k	$(1+k)^{\frac{1}{k}}$
0.1	2.59374
0.01	2.70481
0.001	2.71692
0.0001	2.71815
0.00001	2.71827

f. Based on the table, it appears that the man's balance will approach $\$1000(e)^{0.05}\approx$ $\$1051.27$