Zeroes and factorization of a non-polynomial function

Task

a. Sketch graphs of the functions f and F given by $f(x) = |x|$ and $F(x) = x^2$ for $-2 \leq x \leq 2$.

b. Suppose g is the function given by $g(x) = \frac{f(x)}{x}$ for $x \neq 0$ and G is the function given by $G(x) = \frac{F(x)}{x}$ for $x \neq 0$. Sketch graphs of the functions g and G for $x \neq 0$ and $-2 \leq x \leq 2$.

c. Is there a natural way to define g and G when $x = 0$? Explain.

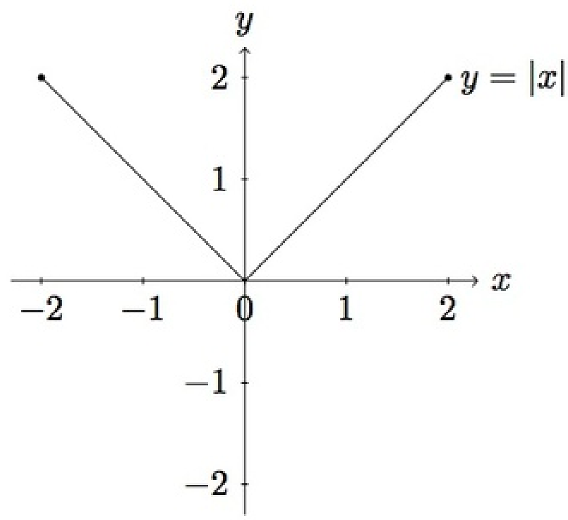
Commentary

For a polynomial function f, if $f(0) = 0$ then the polynomial $f(x)$ is divisible by x. This fact is shown and then generalized in "Zeroes of a quadratic polynomial I, II" and "Zeroes of a general polynomial." Here, divisibility tells us that the quotient $\frac{f(x)}{x}$ will still be a nice function -- indeed, another polynomial, save for the missing point at $x = 0$. The goal of this task is to show via a concrete example that this nice property of polynomials is not shared by all functions. The non-polynomial function F given by $F(x) = |x|$ is a familiar function for which property does not hold: even though $F(0) = 0$, the quotient $\frac{F(x)}{x}$ behaves badly near $x = 0$. Indeed, its graph is broken into two parts which do not connect at $x = 0$.

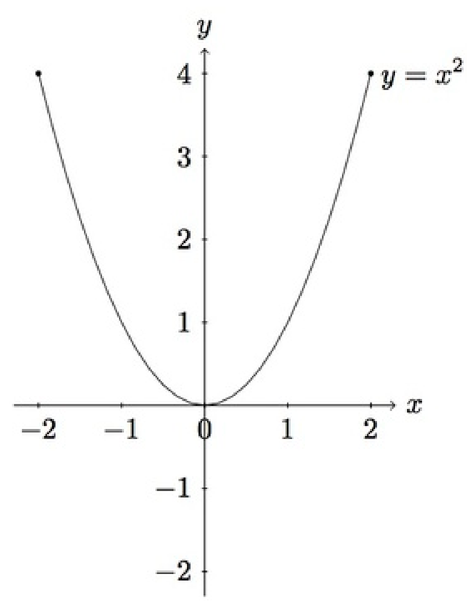
The level of the task is appropriate for assessment but it is principally designed for instructional purposes only. The students may use graphing technology: the focus, however, should be on what happens to the function g when $x = 0$ and the calculator may or may not be of help here (depending on how sophisticated it is!).

Solution

a. Below is a picture of the graph of the equation $y = |x|$ when $-2 \le x \le 2$: it is the graph of the equation $y = x$ when $0 \le x \le 2$ and the graph of the equation $y = -x$ when $-2 \le x \le 0$.



Below is a picture of the graph of the equation $y = x^2$ when $-2 \le x \le 2$:



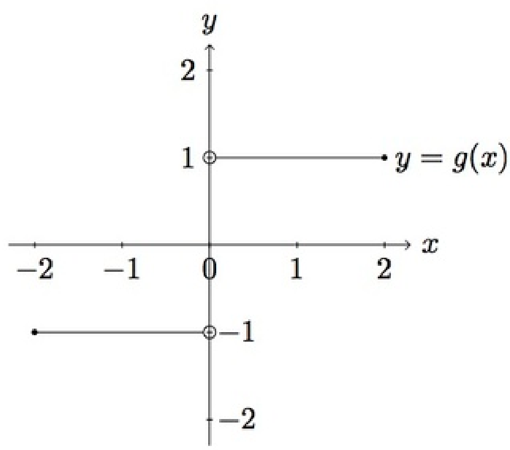
b. If g satisfies $g(x) = \frac{f(x)}{x}$ this means that $g(x) = \frac{|x|}{x}$. When $x > 0$ we have $|x| = x$ and so in this case

$$g(x) = \frac{x}{x} = 1.$$

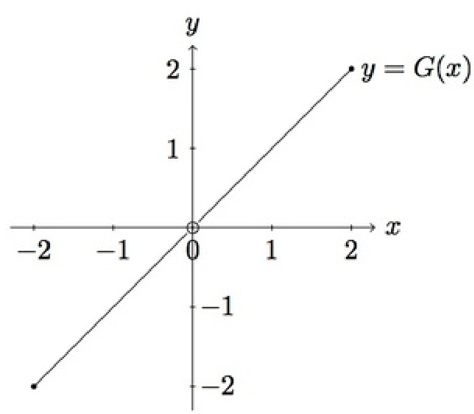
Similarly, if $x < 0$ then $|x| = -x$ and so

$$g(x) = \frac{-x}{x} = -1.$$

Below is a graph of the function $y = g(x)$:



If G is a function which satisfies $G(x) = \frac{F(x)}{x}$ this means that $G(x) = \frac{x^2}{x} = x$. The graph of G is shown below for $-2 \leq x \leq 2$ and $x \neq 0$:



c. The graph of g is broken into two parts which do not come together when $x = 0$. To the right of $x = 0$, g always takes the value 1 while to the left of $x = 0$, g always takes the value -1. Coming from the left g appears as if it will take the value -1 when $x = 0$ while coming from the right it appears as if g will take the value 1 when $x = 0$. There is no "natural" choice for what value g should take when $x = 0$.

Unlike $g(x)$, the function $G(x)$ looks exactly like the function $h(x) = x$ except that the point (0,0) is missing. This makes sense because

$$G(x) = \frac{x^2}{x} = x$$

and the only problem with G is that it has been expressed with an x in the denominator so that it is not defined when $x = 0$. Looking at the graph of G, however, it is natural to define $G(0) = 0$.