Zeroes and factorization of a general polynomial

Suppose $p(x)$ is a polynomial of degree $d > 0$.

a. If $p(0) = 0$, show that $p(x)$ is evenly divisible by x.

b. If $p(1) = 0$, show that $p(x)$ is evenly divisible by $x - 1$.

c. If r is a real number such that $p(r) = 0$, show that $p(x)$ is evenly divisible by $x - r$.

d. Using part (c) show that p can have at most d distinct roots, that is, there can be at most d numbers $r_1, \ldots, r_d$ with $p(r_1) = \cdots = p(r_d) = 0$.

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Commentary

This task builds on ``Zeroes and factorization of a quadratic function" parts I and II. The teacher may wish to recall the result from the first of these tasks, generalized to the polynomials of degree d considered here. This result, the division algorithm applied to polynomials, says that if $p(x)$ is a polynomial of degree d and $l(x)$ is a linear polynomial then

$$p(x) = q(x)l(x) + r$$

where r is the remainder of the division process, a real number in this case.

Although a polynomial of degree d can have at most d roots, it is not easy, once the polynomial is of degree larger than two, to find those roots. In fact, a closed form for the roots of a polynomial function is only available for degrees 2,3, and 4. The level of complexity of these expressions grows rapidly.

This task is related to a very deep theorem in mathematics, the Fundamental Theorem of Algebra, which says that a polynomial of degree d always has exactly d roots, provided complex numbers are allowed as roots and provided roots are counted with the proper ``multiplicity" (for example the polynomial $f(x) = x^2$ has only one root, $0$, but this root occurs with ``multiplicity" 2). The teacher may wish to take this occasion to mention this important theorem. The teacher may also wish to discuss the inductive structure of the argument in part (d): the impact of each root on the factorization of the polynomial $p(x)$ builds upon the factorization found in the previous step.

Solution: 1

a. If 0 is a root of the function p this means that $p(0) = 0$. Since p is a polynomial of degree d it is given by a formula

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0,$$

where $a_d \neq 0$. If we divide $p(x)$ by x, using long division of polynomials, we will find

$$p(x) = q(x)x + s$$

where $q(x)$ is a polynomial of degree $d-1$ and the remainder s is a number. Plugging in $x = 0$ we find

$$\begin{aligned} 0 &= p(0) \\ &= q(0)(1-1) + s \\ &= s. \end{aligned}$$

Thus $s = 0$ and so $p(x) = xq(x)$ and $p(x)$ is evenly divisible by x as desired.
Alternatively, we see from inspection that $p(0) = a_0$ and so a_0 is 0 when $p(0) = 0$. Since all other terms of $p(x)$ have at least one power of x, we can conclude that $p(x)$ is evenly divisible by x. This argument is quicker than the preceding but does not generalize as readily to the other parts of the problem.

b. If 1 is a root of the function p this means that $p(1) = 0$. If we divide $p(x)$ by $x-1$, using long division of polynomials, we will find

$$p(x) = q(x)(x-1) + s$$

where $q(x)$ is a polynomial of degree $d-1$ and the remainder s is a number. Plugging in $x = 1$ we find

$$\begin{aligned} 0 &= p(1) \\ &= q(1)(1-1) + s \\ &= s. \end{aligned}$$

Thus $s = 0$ and so $p(x) = (x-1)q(x)$ and $p(x)$ is evenly divisible by $(x-1)$ as desired.

c. If r is a real number which is a root of p we have $p(r) = 0$. Performing long division, as in part (b), this time dividing $p(x)$ by $x - r$ gives

$$p(x) = a(x)(x-r) + s$$

where $a(x)$ is polynomial of degree $d-1$ and s is a real number. Plugging in r we find

$$p(r) = a(r)(0) + s.$$

Since $p(r) = 0$ we conclude that $s = 0$ and so $x - r$ divides f evenly.

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d. If there were $d+1$ different real numbers $r_1, \ldots, r_{d+1}$ which are all roots of p, we need to apply the argument of part (c) $d+1$ times and will find that this is not possible because it would give too many factors, of the form $x - r_i$, of p. Concretely, applying part (c) to r_1 we find

$$p(x) = p_1(x)(x - r_1)$$

where $p_1(x)$ has degree $d-1$. Now, evaluating at r_2 gives

$$0 = p(r_2) = p_1(r_2)(r_2 - r_1).$$

Since r_1 and r_2 are distinct, this means that $r_2 - r_1 \neq 0$ so we must have $p_1(r_2) = 0$.

Repeating the above argument with $p_1(x)$ in place of $p(x)$ we conclude that

$$p_1(x) = p_2(x)(x - r_2)$$

where $p_2(x)$ is a polynomial of degree $d-2$. So we have

$$p(x) = p_2(x)(x - r_1)(x - r_2).$$

Continuing in the same way we find

$$p(x) = a_d(x - r_1)(x - r_2)\ldots(x - r_d).$$

But now we see that when we plug in r_{d-1} we get

$$p(r_{d+1}) = a_d(r_{d+1} - r_1)(r_{d+1} - r_2)\ldots(r_{d+1} - r_d)$$

and this is not zero because a_d is not zero and r_{d+1} is distinct from the other roots $r_1, \ldots, r_d$.

Part (d) of this problem is closely related to a very important result in mathematics called the Fundamental Theorem of Algebra. This says that a polynomial of degree d has d roots provided that complex numbers are allowed for solutions and provided the roots are counted with the appropriate ``multiplicity." It is interesting to note that limiting the number of possible solutions can be done, as in this task, within the high school curriculum: on the other hand, showing that d solutions to a polynomial of degree d always exist (when counted appropriately) remains to this day a very difficult problem.

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