Course of Antibiotics

Task

Susan has an ear infection. The doctor prescribes a course of antibiotics. Susan is told to take 250 mg doses of the antibiotic regularly every 12 hours for 20 days.

Susan is curious and wants to know how much of the drug will be in her body over the course of the 20 days. She does some research online and finds out that at the end of 12 hours, about 4% of the drug is still in the body.

a. What quantity of the drug is in the body right after the first dose, the second dose, the third dose, the fourth dose?

b. When will the total amount of the antibiotic in Susan's body be the highest? What is that amount?

c. Answer Susan's original question: Describe how much of the drug will be in her body at various points over the course of the 20 days.

Commentary

This task presents a real world application of finite geometric series. The context can lead into several interesting follow-up questions and projects. Many drugs only become effective after the amount in the body builds up to a certain level. This can be modeled very well with geometric series.

The task should be used primarily for instruction, as questions about the context might confuse students attempting the task during an assessment. (For example, the task uses the true hypothesis that the amount of drug continues to decay exponentially, so that 4% of 4% of it persists in the system after 2 hours, but students might need help making that assumption explicit). As an example of classroom use, one could use the task as part of a set of real-life situations to which geometric series apply, or assign the task to cooperative learning groups to further engagement in the mathematical practices that must be brought to bear.

Finally, an opportunity presents itself here to discuss reasonableness of numerical approximations. In the solution to part (b), we encounter numbers like $(.04)^{39}$, which is approximately 3×10^{-55}, a quantity of milligrams so incredibly tiny that it is completely negligible for any realistic purpose. The solution addresses this only briefly, by noting that it is unreasonable to report back the full scale of decimal accuracy the abstract mathematical model predicts.

Solution

a. Let Q_1 be the amount of the drug in the body after the first dose. Since the dose is 250 mg, $Q_1 = 250$. To find Q_2, the amount of the drug in the body after the second dose, we have to find out how much of the first dose is still present and add the new dose. Since after 12 hours about 4% of the drug is still present in the body we get:

$$\begin{aligned} Q_2 &= 0.04Q_1 + 250 \\ &= (0.04)250 + 250 \\ &= 260. \end{aligned}$$

Right after the second dose, 260 mg of the drug are present in the body. We use the same reasoning to find Q_3 :

$$\begin{aligned} Q_3 &= (0.04)Q_2 + 250 \\ &= (0.04)\big((0.04)250 + 250\big) + 250 \\ &= (0.04)^2 250 + (0.04)250 + 250 \\ &= 260.4. \end{aligned}$$

After the third dose, 260.4 mg of the drug are present in the body. We can see a pattern emerging and find

$$\begin{aligned} Q_4 &= (0.04)^3 250 + (0.04)^2 250 + (0.04)250 + 250 \\ &= 260.416. \end{aligned}$$

b. Since every time Susan takes a new dose of the antibiotic, a small part of the previous doses is still present, the total amount of the antibiotic in her body will be the highest right after the last dose, which is the 20th dose. To find out how large that amount is, we have to find Q_{20}.

From computing Q_1, Q_2, Q_3 and Q_4 we saw a pattern of a finite geometric series emerging. So we have

$$\begin{aligned} Q_{20} &= 250 + 250(0.04) + 250(0.04)^2 + \cdots + 250(0.04)^{39} \\ &= 250(1 + 0.04 + 0.04^2 + \cdots + 0.04^{39}) \\ &= 250\frac{1 - 0.04^{40}}{1 - 0.04} \\ &= 260.4167 \end{aligned}$$

So the greatest amount of antibiotics in Susan's body is 260.4167 mg. Note that this number assumes an *exact* value of 4\% for the amount of the drug left over after a 12-hour period: A more reasonable interpretation of the level of accuracy would be to report the solution of 260.4 mg. This level of accuracy renders negligible the contributions of the drug taken longer than 24 hours ago.

c. From the previous two parts of the problem we know that the greatest amount of the antibiotic in Susan's body is about 260.4167 mg and it will occur right after she has taken the last dose. However, we saw that already after the fourth dose she had 260.416 mg of the drug in her system, which is only insignificantly smaller. So we can say that starting on the second day of treatment, twice a day there will be about 260.416 mg of the antibiotic in her body. Over the course of the following 12 hours the amount of drug will decrease to about 4% of the maximum amount, which is 10.4166 mg. Then the cycle will repeat.