Accuracy of Carbon 14 Dating II

Carbon 14 is a form of carbon which decays exponentially over time. The amount of Carbon 14 contained in a preserved plant is modeled by the equation

$$f(t) = 10\left(\frac{1}{2}\right)^{ct}.$$

Time in this equation is measured in years from the moment when the plant dies ($t = 0$) and the amount of Carbon 14 remaining in the preserved plant is measured in micrograms (a microgram is one millionth of a gram). The number c in the exponential measures the exponential rate of decay of Carbon 14.

a. How many micrograms of Carbon 14 are in the plant at the time it died?

b. The best known estimate for the half-life of Carbon 14, that is the amount of time it takes for half of the Carbon 14 to decay, is 5730 ± 40 years. Use this information to calculate the range of possible values for the constant c in the equation for f.

c. Use your answer from part (b) to find the range of years when there is one microgram remaining in the preserved plant.

Commentary

This task is a refinement of ``Carbon 14 dating" which focuses on accuracy. Because radioactive decay is an atomic process modeled by the laws of quantum mechanics, it is not possible to know with certainty when half of a given quantity of Carbon 14 atoms will decay. The range of years 5730 ± 40 gives a certain probability (about 68 percent) that half of the Carbon 14 will decay during this span of years: it is of course possible that the actual half life could be shorter or longer. Each given sample of Carbon 14 would have to be treated individually on an experimental basis and if many experiments were conducted, an expected 68 percent would give a half-life measured between 5690 and 5770 years.

While the mathematical part of this task is suitable for assessment, the context makes it more appropriate for instructional purposes. This type of question is very important in science and it also provides an opportunity to study the very subtle question of how errors behave when applying a function: in some cases the errors can be magnified while in others they are lessened.

Solution: 1

a. The death of the plant corresponds to $t = 0$ so evaluating f when $t = 0$ gives

$$f(0) = 10\left(\frac{1}{2}\right)^{c\cdot 0} = 10$$

since $\left(\frac{1}{2}\right)^0 = 1$. So regardless of what value is assigned to c, the amount of Carbon 14 in the plant at the time of its death is 10 micrograms.

b. We are given $f(t) = 10\left(\frac{1}{2}\right)^{ct}$. When $t = \frac{1}{c}$ we find

$$\begin{aligned} f\left(\frac{1}{c}\right) &= 10\left(\frac{1}{2}\right)^{\frac{c}{c}} \\ &= 10\left(\frac{1}{2}\right) \\ &= 5. \end{aligned}$$

So the half-life of Carbon 14 is $\frac{1}{c}$. We are given that this half-life lies within a 40 year period of 5730 years, that is

$$5690 \le \frac{1}{c} \le 5770.$$

This gives us inequalities for our constant c, namely

$$\frac{1}{5770} \le c \le \frac{1}{5690}.$$

c. There is one microgram of Carbon 14 remaining when

$$f(t) = 1.$$

Using the value $c = \frac{1}{5770}$ this means that

$$10\left(\frac{1}{2}\right)^{\frac{t}{5770}} = 1.$$

Dividing both sides by 10 and taking the logarithm with base 2 of both sides allows us to solve for t:

$$t = -5770 \cdot \log_2 \frac{1}{10}$$

or about $19,170$ years. Using the value $c = \frac{1}{5690}$ gives

$$t = -5690 \cdot \log_2 \frac{1}{10}$$

or about $18,900$ years. So an appropriate way to record the amount of time that it takes for there to be one microgram of Carbon 14 remaining would be between $18,900$ and $19,170$ years. The range of years here is a little under one percent in either direction of $19,035$ years.