Ice Cream

Task

After a container of ice cream has been sitting in a room for t minutes, its temperature in degrees Fahrenheit is:

$$a - b2^{-t} + b$$

where a and b are positive constants. Write this expression in a form that

a. Shows that the temperature is always less than $a + b$.

b. Shows that the temperature is never less than a.

Commentary

This task illustrates the process of rearranging the terms of an expression to real different aspects about the quantity it represents. Students are provided with an expression giving the temperature of a container at a time t, and have to use simple inequalities (e.g., that $2^t > 0$ for all t) to reduce the complexity of an expression to form where bounds on the temperature of a container of ice cream are made apparent.

Solution

a. To begin, we can first rearrange this expression into:

$$(a+b)-b2^{-t}$$

We can now see that we have an $(a+b)$ together on the left, and our last term is $b2^{-t}$, and this term will dictate if the temperature is greater or less than $(a+b)$. Since b is a positive constant, and since 2^{-t} is positive regardless of the value of t, we know that $b2^{-t}$, is positive. So, we have:

$$a+b-b2^{-t}<a+b$$

b. We can rearrange the expression in the following way:

$$a+b-b2^{-t}=a+b\left(1-\frac{1}{2^t}\right).$$

We see that now the term $b\left(1-\frac{1}{2^t}\right)$ is going to dictate if the temperature is greater or less than a. Since t is the number of minutes that the ice cream has been sitting in the room, we know that t will always be greater than zero. Therefore, $2^t>1$, so $\frac{1}{2^t}<1$, so $\left(1-\frac{1}{2^t}\right)>0$. From this we can conclude that since $b>0$, we have $b\left(1-\frac{1}{2^t}\right)>0$. Therefore,

$$a+b\left(1-\frac{1}{2^t}\right)>a$$