Computations with Complex Numbers

Rewrite each of the following expressions involving complex numbers in the form $a + bi$ where a and b are real numbers.

a. $$(3+2i)(2-5i)$$

b. $$(5+4i)(17-13i)-(5+3i)(17-13i)$$

c. $$\left(\frac{5}{2}+\frac{7i}{2}\right)^2-\left(\frac{5}{2}+\frac{i}{2}\right)^2$$

d. $$(1+i)(13-4i)(1-i)$$

e. $$1+i+i^2+i^3$$

Commentary

This task asks students to perform computations involving complex numbers using the properties of operations and the fact that $i^2 = -1$. Students can complete the task provided that they know this fact and can apply it to find that $i^3 = -i$. However, students who pay attention to the structure of each expression and use properties of operations will be able to avoid or shorten some tedious calculations.

A teacher who uses this problem as a classroom task should keep track of students' solution approaches and ask students to present different solutions to the same problem.

Solution: Solution

a. We have

$$\begin{aligned}(3+2i)(2-5i) &= 3(2-5i)+2i(2-5i)\\ &= 6-15i+4i-10i^2\\ &= 6-15i+4i+10\\ &= 16-11i.\end{aligned}$$

Note that we have used the fact that $i^2=-1$ to write $-10i^2=10$.

b. We can evaluate this expression by computing each product of binomials separately and then subtracting. However, we can make our job easier by noticing that each product contains a factor (17 - 13i), and factoring this out to obtain

$$\begin{aligned}(5+4i)(17-13i)-(5+3i)(17-13i) &= ((5+4i)-(5+3i))(17-13i)\\ &= i(17-13i)\\ &= 17i-13i^2\\ &= 13+17i.\end{aligned}$$

c. Again, it is possible to square each binomial separately and then subtract, but since we have an expression of the form x^2-y^2, let's try factoring the given expression as a difference of squares:

$$\begin{aligned}\left(\frac{5}{2}+\frac{7i}{2}\right)^2-\left(\frac{5}{2}+\frac{i}{2}\right)^2 &= \left(\left(\frac{5}{2}+\frac{7i}{2}\right)-\left(\frac{5}{2}+\frac{i}{2}\right)\right)\left(\left(\frac{5}{2}+\frac{7i}{2}\right)+\left(\frac{5}{2}+\frac{i}{2}\right)\right)\\ &= 3i(5+4i)\\ &= 15i+12i^2\\ &= -12+15i\end{aligned}$$

d. By the commutative and associative properties, we can pair and multiply the factors of this expression in whatever order we want. Since we see two complex conjugates - namely, (1 + i) and (1 - i) - we will try pairing and multiplying those first:

$$\begin{aligned}(1+i)(13-4i)(1-i) &= (1+i)(1-i)(13-4i)\\ &= (1-i^2)(13-4i)\\ &= 2(13-4i)\\ &= 26-8i.\end{aligned}$$

e. We know that $i^2=-1$, and $i^3=i^2\cdot i=-i$. So

$$1+i+i^2+i^3=1+i+(-1)+(-i)=0.$$